The *Numeracy Development and Intervention Guide* was written by Dr. Kenneth E. Vos, professor of education at St. Catherine University, St. Paul, Minnesota. With his strong ties to practitioners and classrooms, Professor Vos has crafted a guide that seamlessly blends research with practice. For more than 35 years, he has been preparing undergraduate and graduate students to teach mathematics, and in tandem working with classroom teachers. He has directed numerous programs and institutes for teachers on mathematics, best practices in the mathematics classroom, and assessment. The advice in this guide reflects the author’s deep respect for the daily work of teachers. Dr. Kenneth E. Vos holds a Ph.D. in mathematics education from the University of Minnesota.
# Contents

## Overview

- Goals ....................................................... 1
- Structure of This Guide ................................. 3
- Core Research ............................................. 4
- Principles ..................................................... 6

## 1 Basic-Level Fraction Numbers ................................................................. 17

- Fraction Number Strategies ................................. 18
- Decimal Number Strategies ................................ 22
- Percent Strategies ........................................... 24
- Informal Diagnostic Strategies ............................ 24
- Common Errors ............................................. 25
- Tips for Success ............................................. 27
- Protocols ....................................................... 28

## 2 Developing-Level Fraction Numbers ......................................................... 57

- Fraction Number Strategies ................................ 59
- Decimal Number Strategies ................................ 63
- Percent Strategies ........................................... 69
- Informal Diagnostic Strategies ............................ 72
- Common Errors ............................................. 73
- Tips for Success ............................................. 75
- Protocols ....................................................... 76

## 3 Advanced-Level Fraction Numbers ......................................................... 105

- Fraction Number Strategies ................................ 107
- Decimal Number Strategies ................................ 112
- Percent Strategies ........................................... 115
- Informal Diagnostic Strategies ............................ 118
- Common Errors ............................................. 119
- Tips for Success ............................................. 121
- Protocols ....................................................... 122

## Appendix  ........................................................................................................ 143

- Reproducible Forms ......................................... 144
- References ...................................................... 175
Overview

“Fractions are just weird. They don’t act like numbers.”
—fifth-grade student

Fraction numbers can be frustrating. This young student expresses a common feeling among learners beginning to study fraction numbers, decimal numbers, and percents. Confusion surrounds these three essential math topics. Although fraction numbers appear in all levels of mathematics study, learners often never grasp the conceptual basis underlying them. (The intense pressure to teach many other important topics may leave teachers with little time for direct instruction in fraction numbers, decimal numbers, and percents.) And yet a thorough and complete understanding of fraction numbers, including their interrelationship with decimal numbers and percents, is crucial to students’ success in advanced mathematics study. This teacher’s guide provides a research- and practice-based model to help you lead your students to a solid fraction number sense.

A learner must acquire a solid number sense in order to make judgments about numerical results and explore the relationships among numbers. Learners who have acquired number sense exhibit an understanding of different number meanings, how numbers are related, how they differ in magnitude, and the effect of number operations. Fraction number sense requires these same components of understanding. (In most contexts, the terms fractions and fraction numbers include decimal numbers and percents; fraction number sense also encompasses these forms as well.) Through a carefully crafted sequence of instructional strategies and activities, teachers can help students develop their fraction number sense. This guide offers such a plan. Along with these instructional strategies, informal diagnostic tools are suggested to help teachers determine the appropriate interventions when misunderstandings arise. We are confident that the thoughtful sequence of action plans, based on developmentally reasonable expectations, will enable the learner to gain a solid foundational understanding of fraction numbers, decimal numbers, and percents.

Goals

The goals of this guide are twofold:

1. To provide teachers with a context (within the mathematics curriculum) to help their students explore the concept of fraction numbers and acquire basic fraction number sense
   Learners must explore fraction concepts before they can confidently
compare fraction numbers, decimal numbers, and percents. Understanding the interrelationship of these three modalities is key to acquiring fraction number sense. Grasping these relationships entails seeing just how fraction numbers are similar to, yet different from, whole numbers.

2. **To present a set of intervention strategies and techniques to help students who are having trouble with fraction concepts**

Having a toolkit of intervention strategies and techniques is paramount for developing learners’ confidence as they study the complex nature of fraction numbers. The toolkit provided in this guide addresses fraction numbers in depth; it also includes decimal and percent activities.

To achieve the goals listed above, this guide does the following:

- **It offers teachers a thorough investigation of the fundamental concepts of fraction number sense.**
  A firm foundation in fraction number sense is necessary for proficiency in advanced mathematics courses as well as in daily life. On a fairly regular basis, we need to estimate with whole numbers in our day-to-day life. We often also find that we need to be able to estimate with more precision than whole numbers allow. Fraction numbers fill the need for more precise values in daily life. Many recipes, for example, require fractional accuracy. Suppose you are making a loaf of bread; the recipe may call for 4 \( \frac{1}{2} \) cups of flour. Confidence in using fraction numbers is necessary in measurement contexts. This guide is designed for teaching fraction numbers, decimal numbers, and percents to learners in late second grade through tenth grade. You will find strategies and techniques for all these ages—late primary, intermediate, middle/junior high, and early high school—in the guide.

- **It explains informal diagnostic strategies for detecting fraction misunderstandings and confusion in a student’s fraction number sense.**
  A brief plan of informal diagnostic strategies for specific situations is given in this guide. These strategies are designed for a variety of contexts, including small-group settings, individual tutoring sessions, or at-home review with a parent or guardian at the dining room table. A diagnostic cycle is included to guide the process from beginning to completion.

- **It offers constructed protocols that are ready for teacher or parent use.**
  The protocols in this guide are basic frameworks that include a specific topic, a realistic time period in which to complete the activity, a sequence of actions, modifications for different types of learners, and an evaluation strategy. These protocols can be used by a classroom teacher for individual or small group intervention, by a parent or guardian at home to supplement
the intervention given by the classroom teacher, and by an instructional aide as extra support for the classroom teacher’s intervention.

- **It presents activities and games that can enhance the understanding of fraction number sense.** Some of the protocols in this guide take the form of group activities and games designed to enhance students’ fraction number sense. As learning tools, activities and games must reflect accurate mathematics, balance competition and cooperation, align with best practices in instruction, and be fun! The games and activities in this guide are all of those things, as well as being appropriate for all our target-age learners.

**Structure of This Guide**

Three sections—Basic Level, Developing Level, and Advanced Level—reflect the different developmental stages of acquiring fraction number sense. Still, the three sections present overlapping content. Each section is structured around three guiding principles appropriate for that learning level and ends with Tips for Success, a helpful summary of advice from classroom teachers.

The Basic-Level section focuses on primary-age learners’ needs. It begins with a brief overview of the development of fraction number sense and the relationships among whole numbers, fraction numbers, decimal numbers, and percents. The Basic-Level section emphasizes fraction numbers, with limited focus on decimal numbers and percents. It also includes informal diagnostic strategies for detecting when learners are misunderstanding fraction numbers, a brief explanation of intervention strategies to correct these misunderstandings, and a set of 12 intervention protocols.

The Developing-Level section focuses on intermediate and middle/junior high school learners. It briefly summarizes the Basic-Level section, but includes a more advanced perspective on fraction number sense and gives more weight to decimal numbers and percents. The interrelationship of fraction numbers, decimal numbers, and percents is explored in detail. The power of estimation and benchmarking to 0, \(\frac{1}{2}\), or 1 for various fraction values is explored. Like the Basic-Level section, this one includes informal diagnostic strategies for

---

**Where to Begin?**

If you are using this guide in an intervention setting, you may be working with learners who are several grade levels behind in their understanding of fraction numbers, decimal numbers, and percents. Therefore, older students may need to review concepts at the basic level before working in the developing or advanced level. In an intervention setting, use the diagnostic strategies, list of common errors, and protocols in each section to determine the appropriate level in which to begin each of your students.
detecting when learners are misunderstanding a concept, a brief explanation of intervention strategies to correct these misunderstandings, and a set of 12 protocols for intervention.

The third section, Advanced-Level Fraction Numbers, focuses on early high school learners. After a brief summary of the Developing-Level section, it expands to cover precision and significant digits for decimal numbers and percents. An informal diagnostic strategy is included to help teachers identify specific errors regarding fraction numbers, decimal numbers, and percents. Addition and subtraction of fraction numbers is explored, with the main emphasis on fraction numbers with common denominators. A key focus for this level is on estimating and benchmarking to 1 in situations with addition and subtraction of fraction numbers with unlike denominators. A set of eight protocols for intervention is also included.

The appendix provides reproducible pages of materials needed for some of the protocols; these instructional aids include Fraction Number Cards, Fraction Circles, 10 x 10 square grids, as well as game boards and recording sheets for specific activities. In addition, some protocols require readily available materials such as number cubes and pattern blocks. Each protocol identifies the materials needed to successfully complete the activity.

This guide is designed for classroom teachers or parents of all ages of learners. All sections contain information and strategies to help you lead a learner to a well-developed fraction number sense, regardless of the age of that learner. Use this resource as your guide to appropriate interventions to teach fraction numbers and their close relatives, decimal numbers and percents.

Core Research

The dismal state of mathematical understanding among American students has been well documented over the last 25 years. American students’ achievement lags far behind their counterparts’ in other parts of the world, particularly Asia and a few European countries (Gonzales, et al., 2009). According to a recent What Works Clearinghouse practice guide (Siegler et al., 2010), “poor understanding of fractions is a critical aspect of this inadequate mathematics knowledge” (p. 6). A hazy grasp of fraction numbers causes problems for students as they move on to algebra and advanced analysis, for which a deep understanding of fraction numbers is required (National Mathematics Advisory Panel, 2008). In the National Survey of Algebra Teachers, Algebra I teachers rated their students’ preparation in fraction numbers and decimal numbers as inadequate, and identified this knowledge gap as one of the most significant weaknesses in the learners’ lack of success in beginning algebra (Hoffer et al., 2007).

Recent national assessment programs show that students at different grade levels suffer from this weak understanding of fraction numbers. The 2004
National Assessment of Educational Progress (NAEP) found that only 50% of eighth grade learners could order three fraction numbers from smallest to largest (National Council of Teachers of Mathematics [NCTM], 2007b). Also, an analysis of NAEP results showed that fewer than 30% of 17-year-olds could correctly express 0.029 as \( \frac{29}{1000} \) (Kloosterman, 2010). Learners in the fifth and sixth grades were not proficient in the concept of decimal magnitude: when asked to choose the larger decimal of 0.274 and 0.83, most chose 0.274 (Rittle-Johnson et al., 2001). These results tell us that fraction numbers are one of the weak areas of mathematics instruction in the United States.

A high percentage of U.S. students lack conceptual understanding of fraction numbers, even after studying them for several years; this, in turn, limits students' ability to solve problems and to learn and apply computational procedures involving fraction numbers (Siegler et al., 2010). Further research has identified the following common conceptual errors, which have direct implications for mathematics instruction.

Students may

- not view fractions as numbers at all, but rather as meaningless symbols that need to be manipulated in arbitrary ways (Siegler et al., 2010).
- focus on numerators and denominators as separate numbers rather than thinking of the fraction as a single number (Siegler et al., 2010).
- confuse properties of fraction numbers with those of whole numbers (Siegler et al., 2010). For example, some high school learners believe there is no number between \( \frac{5}{7} \) and \( \frac{6}{7} \), just as there is no whole number between 5 and 6 (Vamvakoussi & Vosniadou, 2004).

A careful intervention strategy around fraction numbers, decimal numbers, and percents must take into account these findings and their implications. The foundation for a deep understanding of fraction numbers begins with the simple knowledge of sharing and proportionality that young children bring to school with them (Davis & Pepper, 1992). The National Council of Teachers of Mathematics (2000) recommends that teachers use activities that build new knowledge from learners' prior experience and knowledge. Upon this early foundation, then, teachers can next build learners' understanding of these specific concepts:

- Common fraction numbers, decimal numbers, and percents are equivalent ways of expressing the same number: \( \frac{42}{100} = 0.42 = 42\% \) (Seigler et al., 2010).
- Whole numbers are a subset of rational numbers (Seigler et al., 2010).
- Any fraction number can be expressed in an infinite number of equivalent ways: \( \frac{3}{4} = \frac{6}{8} = 0.75 = 75\% \) (Seigler et al., 2010).
A caveat must be expressed about the research on fraction knowledge and teaching. Many of the strategies used to teach fraction numbers are based on best practices and experience, rather than research-based studies. In fact, “…far less research is available on fractions than on development of skills and concepts regarding whole numbers” (Siegler et al., 2010, p. 10). A National Council of Teachers of Mathematics survey of mathematics education research published between 2000 and 2007 showed 109 citations on whole numbers and only nine related to fraction numbers (NCTM, 2007a). “High-quality studies testing the effectiveness of specific instructional techniques with fractions were especially scarce,” Seigler et al. (2010, p. 10) noted.

To augment the research base when writing this guide, the author conducted three structured focus groups, one each with primary, middle school, and high school classroom teachers. Eight teachers from each level, who taught in both urban and suburban settings, answered the same four questions. Every teacher was given time to describe his or her experience and discuss the questions. Many of the suggestions, techniques, and instructional and assessment strategies in this guide therefore reflect teacher best practices culled from the focus groups, as well as from the author’s experience working with teachers and from recent textbook information.

**Principles**

Each section of this guide presents three principles. These principles are gleaned from the findings discussed in the Core Research section; from the author’s own experience teaching fraction numbers, decimal numbers, and percents; and, most importantly, from second-grade to high school mathematics teachers who participated in the focus groups. The principles are meant as guideposts to help you successfully teach fraction numbers. Organizing this guide around these principles offers a concise approach to a complex and often confusing subject area.

**Basic Level**

**Basic-Level Principle 1**

Build upon knowledge students already possess: introduce fraction numbers by having students explore how to share a set of objects equally among a group.

Many students come to preschool or kindergarten having had experiences sharing. They may have “divided,” or shared, cookies or toys with siblings or friends. As a result, these students already have an informal understanding of dividing. Build on these experiences for further exploration of fraction numbers. This will give learners a firm foundation for building their fraction number sense.
Sample Problem
A friend came over to your house to play. You have four toys that you want to play with together. How many toys would you and your friend each get to play with so that you both have the same number of toys?

Each of you will get two toys to play with for a while.

Basic-Level Principle 2
Introduce division in the early grades: present sharing and the concept of division simultaneously.

The concepts of division and sharing are closely aligned. Interestingly, of the four basic operations of addition, subtraction, multiplication, and division, a young child is most familiar with division, through the practice of sharing. Within a child's world the concept of division occurs more frequently than the concepts of increase (adding or multiplying) or decrease (subtracting). Even though it may seem counter-intuitive, the simultaneous presentation of sharing and division is appropriate for young children.

Sample Problem
A candy bar is shared equally among 3 children. How much candy bar does each child get?

Each child gets one of the three parts or $\frac{1}{3}$ of the candy bar.
Basic-Level Principle 3
Use equal sharing activities to introduce the concept of size in fraction numbers.

When fraction numbers are introduced in later grades (usually in fifth grade in tandem with multiplication), learners may see them as mere abstractions. This obscures the fact that fraction numbers denote size; some fraction numbers are bigger than others. Learners easily misunderstand this aspect of fraction numbers. They may focus on the denominator only, and assume that \(\frac{1}{4}\) is larger than \(\frac{2}{3}\) because 4 is larger than 3. They may also assume that all fractions are ways of saying \(\frac{1}{2}\), since many of their sharing experiences involve one part divided into two.

Sample Problem
A personal-sized cheese pizza is cut into 4 equal pieces. A pepperoni pizza of the same size is cut into 3 pieces.

You take 1 piece from the cheese pizza (\(\frac{1}{4}\) of the pizza). Your friend takes 2 pieces from the pepperoni pizza (\(\frac{2}{3}\) of the pizza). Who has the greater amount of pizza to eat? You, with \(\frac{1}{4}\), or your friend, with \(\frac{2}{3}\)? Who has the smaller amount of pizza to eat?
Developing Level

**Developing-Level Principle 1**
Emphasize that fractions are numbers, and they have magnitude.

Many learners at this level are quite confident in their knowledge of whole numbers and their distinct values. They can easily order whole numbers from smallest to largest, determine equivalent values from number sentences and perform basic arithmetic operations with whole numbers. However, these same learners may be confused when told that fraction numbers also have distinct values. To be successful working with fractions, learners must grasp this key understanding that fractions are numbers, too. They denote different amounts, or different magnitude. So, $\frac{1}{2}$ has greater magnitude than $\frac{1}{3}$, and $\frac{1}{3}$ has greater magnitude than $\frac{1}{4}$ and $\frac{1}{5}$.

**Sample Problem**
Order the following whole numbers from smallest to largest.

8  3  16  21  13  9

(Answer: 3  8  9  13  16  21)

Now order the following fraction numbers from smallest to largest.

$\frac{1}{2}$  $\frac{1}{4}$  $\frac{1}{3}$  $\frac{2}{3}$  $\frac{3}{4}$

(Answer: $\frac{1}{4}$  $\frac{1}{3}$  $\frac{1}{2}$  $\frac{2}{3}$  $\frac{3}{4}$)

**Developing-Level Principle 2**
Explore equal sharing activities to develop learners’ understanding of fractional equivalence.

Since sharing is the foundation for building fraction understanding, it should be applied when teaching the concept of equivalent fraction numbers. Equivalence means to have the same value. In its simple form, equivalence means expressing a fraction number in different terms, just as we can express whole numbers in different terms. The number 3 can be expressed as $4 - 1, 1 + 2, 1 + 1 + 1$, or $\frac{6}{2}$. Likewise, the fraction number $\frac{1}{3}$ can be expressed as $\frac{2}{6}, \frac{3}{9}, \frac{4}{12}$, and $\frac{5}{15}$, to name a few equivalents.
Sample Problem

It’s your birthday! You have a cake to share with your friends. There are 4 of you. You cut it into equal pieces to share with everyone.

Each piece is \(\frac{1}{4}\) of the total cake. The pieces are big, and much too much to eat all at once. You and your friends decide to cut your pieces in half, so you can eat some now and the rest later.

Now cut these 4 equal pieces in half.

Each piece is \(\frac{1}{8}\) of the total cake.

How much of the cake did each of you get? \(\frac{1}{4}\) or \(\frac{2}{8}\)? Both! These fraction numbers name the same amount. So, \(\frac{1}{4}\) and \(\frac{2}{8}\) are equivalent fraction numbers.

Developing-Level Principle 3

Use a number line to represent fraction and decimal numbers; this technique builds appropriate understanding.

Research findings support using a number line to help learners acquire a deep understanding of fraction and decimal numbers within a mathematical context. The number line fits into the instructional strategy of presenting a concept first through the concrete (hands-on) mode, moving next to the pictorial mode, and finally exploring the concept in the abstract mode.
A number line can be a concrete strategy if used as a hands-on activity, but most of the time it is a pictorial strategy, being represented as a diagram or picture. The number line is a simple but powerful strategy for teaching many aspects of fraction numbers.

Sample Problem

![Number Line Diagram]

Using this number line, estimate and mark where $\frac{1}{4}$ and $\frac{3}{4}$ are located.

Now estimate and mark where $\frac{1}{3}$ and $\frac{2}{3}$ are located.

The estimates are just that: estimates. However, the relative order and magnitude are represented on the number line. This visual clearly suggests the order and magnitude of fraction numbers. The same procedure can be used with decimal numbers.

Using this number line, estimate and mark where 0.25, 0.50, and 0.75 are located.

Then estimate and mark 0.40 and 0.80 on the same number line.
Advanced Level

Advanced-Level Principle 1

Emphasize the concepts of significant digits and precision: learners must have an advanced number sense for a clear understanding of these complex topics.

A number can be represented by many expressions. For example, \( \frac{1}{2} = 0.5 = 50\% \). However, not all fraction numbers are as easy to express as a decimal number or percent. Consider \( \frac{1}{3} \), which is often expressed as 0.33—but actually, it is a repeating decimal and has an infinite number of 3s in its decimal expansion. So, 33\% is not an accurate value for \( \frac{1}{3} \) of a value (33 \( \frac{1}{3} \) \% is more accurate).

The concept of significant digits comes into the picture when teaching more complex decimal and fraction numbers. In decimal numbers, the significant digits include the digits beginning (from the left-most digit) with the first nonzero digit and ending with the last digit that appears after the decimal point. For example, the numbers 682, 68.2, and 0.00682 all have three significant digits: 6, 8, and 2. The numbers 6820, 682.0, 68.20, and 0.006820 all have four significant digits, namely 6, 8, 2, and 0. Precision with decimal values is important when using measurement values in science or finance.

Sample Problem

An ad in the newspaper stated that a cell phone was on sale for \( \frac{1}{3} \) off the regular price. The regular price was $68.00. How did the electronic store determine the sale price?

Which sale price do you think should have been listed on the ad?
Overview

• \( \frac{1}{3} \) of \$68 or \( \frac{68}{3} \) is \$22.6666 repeating. Using significant digits for money and rounding, we get \$22.67. \$68 - \$22.67 = \$45.33

• 33\% of \$68 or \( 68 \times 0.33 \) is \$22.44. \$68 - \$22.44 = \$45.56.

• 33 \( \frac{1}{3} \)\% of \$68 or \( 68 \times 0.3333 \) is \$22.6644. Using significant digits for money and rounding, we get \$22.66. \$68 - \$22.66 = \$45.34

Note the results for 33\% and 33 \( \frac{1}{3} \)\% differ by more than 20 cents! And, the result for \( \frac{1}{3} \)\% off is not the same as for 33 \( \frac{1}{3} \)\% off—though they are very close.

The store should have the sale price as \$45.33 to reflect \( \frac{1}{3} \) off.

**Advanced-Level Principle 2**

Teach estimating and benchmarking; they are powerful tools to help students understand fraction numbers, decimal numbers, and percents.

The facility to estimate whole numbers is crucial to success in many mathematical situations. So is the facility to estimate fraction numbers, decimal numbers, and percents. Exploring benchmarks is a solid strategy to help learners build confidence in estimating fraction numbers, decimal numbers, and percents. These benchmarks do not need to be complex values; students can learn a simple set of well-understood values. For example, the fractional benchmarks could be \( \frac{1}{4} \), \( \frac{1}{2} \), \( \frac{3}{4} \), and maybe \( \frac{1}{3} \) and \( \frac{2}{3} \), as well. The expressions that match these benchmarks in decimal numbers could also be used (0.25, 0.50, 0.75, and 0.33, 0.67). The same benchmarks could be used for percents, too (25\%, 50\%, 75\%, and maybe 33\%, 67\%).

**Sample Problem**

Estimate the location of the following decimal numbers on the number line.

0.42   0.59   0.15   0.89
**Advanced-Level Principle 3**

Ensure a deep understanding of fraction numbers and their attributes before introducing addition and subtraction with fraction numbers.

Before attempting to add or subtract fraction numbers, learners must be confident in their knowledge of fraction numbers and their attributes. A hurried approach to these operations will result in a host of misunderstandings. Mastery of many complex concepts is required even to add a simple combination of two fraction numbers with like denominators.

**Sample Problem**

Add these two fraction numbers.

\[
\frac{5}{9} + \frac{2}{9} = ???
\]

Answers learners might give include \(\frac{7}{18}\), \(\frac{25}{9}\), or \(\frac{7}{9}\).

- One learner may think that completing the operation of addition involves adding the numerators and adding the denominators, resulting in \(\frac{7}{18}\). This is not correct.
- Another learner may think that adding all the values together \((5 + 9 + 2 + 9 = 25)\) and keeping the denominator will give the correct result: \(\frac{25}{9}\). But this is not correct.
- However, a learner with fraction number sense will add the numerators of fraction numbers with a common denominator, resulting in \(\frac{7}{9}\). This is the correct answer.

The interventions in this guide are organized around these nine general principles.
Basic-Level Fraction Numbers

<table>
<thead>
<tr>
<th>Page</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>Fraction Number Strategies</td>
</tr>
<tr>
<td>22</td>
<td>Decimal Number Strategies</td>
</tr>
<tr>
<td>24</td>
<td>Percent Strategies</td>
</tr>
<tr>
<td>24</td>
<td>Informal Diagnostic Strategies</td>
</tr>
<tr>
<td>25</td>
<td>Common Errors</td>
</tr>
<tr>
<td>27</td>
<td>Tips for Success</td>
</tr>
<tr>
<td>28</td>
<td>Protocols</td>
</tr>
<tr>
<td>30</td>
<td>Make a Whole</td>
</tr>
<tr>
<td>32</td>
<td>Make a Half</td>
</tr>
<tr>
<td>34</td>
<td>In-Between Numbers</td>
</tr>
<tr>
<td>36</td>
<td>Comparing Fraction Numbers</td>
</tr>
<tr>
<td>38</td>
<td>$&lt;$ = $&gt;$ (Less Than, Equal To, Greater Than)</td>
</tr>
<tr>
<td>40</td>
<td>Fractions Using Interlocking Circles</td>
</tr>
<tr>
<td>42</td>
<td>Fraction Block Covers</td>
</tr>
<tr>
<td>44</td>
<td>Block Trades</td>
</tr>
<tr>
<td>46</td>
<td>Close Is Just Fine</td>
</tr>
<tr>
<td>48</td>
<td>Grid Lock</td>
</tr>
<tr>
<td>50</td>
<td>Shade the Percents</td>
</tr>
<tr>
<td>52</td>
<td>Match Them All</td>
</tr>
</tbody>
</table>
Basic-Level Fraction Numbers

Before they even start kindergarten or first grade, young learners will have had many experiences involving fraction numbers in their day-to-day lives. Most of these experiences have to do with sharing; in their social interactions, young children learn to share not only things, such as toys and food, but also time (“you can use the scooter for a half hour, then it’s your brother’s turn for a half hour”) and space (“move over and let your sister have half of the couch”). A solid foundation for understanding fraction numbers can be built on these early personal experiences. This section gives a brief overview of beginning fraction number sense, including simple decimal numbers and percents.

Fraction instruction can begin very early in a child’s formal schooling. While you might not introduce fraction numbers in their numerical form with the — symbol at this stage, you can emphasize the words that express fraction numbers. For example, if there are three toys in the play area and three children, you can tell learners that each child has one-third of the toys to play with (if they are sharing toys equally). In this way, learners associate the verbal term one-third with the experience. It is crucial that the learners grasp the fundamental concept of fraction numbers before moving to more abstract representations of fraction numbers. This is what we mean by developing learners’ fraction number sense.

The Basic-Level section is organized into six major topics: beginning fraction number strategies, beginning decimal number strategies, beginning percent strategies, informal diagnostic strategies for identifying learner misunderstandings, common errors, and tips for success. The strategies are not exhaustive or definitive but provide sample approaches that have proven successful with beginning learners. The Tips for Success section at the end...
offers a quick reference list of best practices in teaching fraction numbers at the beginning level.

After the discussion of teaching strategies, you’ll find 12 protocols, or activities, appropriate for beginning learners. Nine focus on fraction numbers; the other three address decimal numbers, percents, and the relationships among fraction numbers, decimal numbers, and percents. Consider these protocols as templates with a specific focus that can be modified as needed. They are not meant to provide complete lesson plans or curriculum units. While designed primarily for small- and whole-group instruction, the protocols can also be used for individual tutoring by an education aide or parent.

**Fraction Number Strategies**

The word fraction derives from the Latin word meaning to break. Fraction numbers are numbers representing objects that have been “broken” into parts. The formal definition is as follows: fraction numbers are rational numbers; they can be written as \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \) does not equal zero. A fraction number with a numerator smaller than its denominator (e.g., \( \frac{2}{3} \) or \( \frac{3}{4} \)) is called a proper fraction number. A fraction number is called improper when the numerator is greater than or equal to the denominator, as in \( \frac{3}{2}, \frac{5}{4}, \text{and} \frac{6}{6} \). A fraction number whose numerator is 1 is called a unit fraction number: \( \frac{1}{4}, \frac{1}{3} \), for example.

Becoming comfortable with fraction numbers takes time. We might call this process developing number sense with fractions, or developing one's fraction number sense. That is, learners need to see fractions as numbers first, and from this foundation they can build their conceptual understanding of fraction numbers. Part of that conceptual grasp is the close relationship of fraction numbers to sharing or division.

**Basic-Level Principle 1**

Build upon knowledge students already possess: introduce fraction numbers by having students explore how to share a set of objects equally among a group.

When beginning formal instruction with fraction numbers, it is wise to stay away from symbols; instead, use words to describe them (e.g., Felicia ate one-half of the candy bar). This moves the focus from “how many” to “what is being considered.” An error in conception can arise, however, with certain statements. For instance, saying “cut the candy bar in half” can obscure the fact that half is a number just as 23 is a number. The learner may think that half is an action, not the name of a part.
Begin your instructional sequence with the concept of half. It’s a good place to start because a) halving things is a common experience for young children, b) sharing for two is easily demonstrated, and c) cutting (dividing) a whole object into two parts relates to the concepts of same size and equal parts. The general instructional sequence of teaching modes would be **concrete** (hands-on) to **pictorial** to **abstract** or symbols.

Be cautious about introducing formal fraction terms in this early instructional sequence. For example, use the terms *top number* and *bottom number* when introducing fraction numbers at this stage. The words *numerator* and *denominator* might shift students’ attention from the number concept to understanding the new words. You can introduce the formal names later.

When introducing the symbols used in fraction numbers, explore the bottom number first, before the top number.

**Example**

You have a large candy bar already marked into 4 equal parts. You take 3 of the equal parts and give them away. What part of the whole candy bar was given away?

![Candy Bar](image)

Start with the bottom number. The number of equal parts of the whole candy bar is 4. So, you can start to write the fraction like this: \( \frac{3}{4} \).

Now, you give away 3 of the parts. The number you give away becomes the top number of the fraction. So, the fraction number is \( \frac{3}{4} \).

You can explain to students that in fraction numbers, the top number *counts the parts* and the bottom number *tells what size parts* are being counted. Using area or region models is helpful when introducing fraction number sense, since young learners easily understand the part-whole mode. Later, learners can explore discrete objects as parts of a set.

It is important for students to grasp the concept of the unit within the context of fraction numbers. Unfortunately, this is one of most confusing aspects of fraction
numbers for many learners. That is why we recommend teachers spend ample time exploring the meaning of a whole, or unit, and provide plenty of practice for learners. Once they have developed this understanding, though, learners can build a fraction by thinking about the unit.

Example
How many equal-sized parts are there in the whole square? Record that number and make a roof over it. Now count the shaded parts and place that number on top of the roof.

\[
\begin{array}{c|c|c|c|c}
\hline
& & & & \\
\hline
\text{roof} & 4 & 3 & & \\
\hline
\end{array}
\]

The unit, or whole, can be complex and sometimes confusing to learners. A fraction number can represent a situation with one whole, that is, a single object (e.g., a watermelon); but the unit can also be a group of things (e.g., a carton of eggs, where 12 eggs make up the unit). Understanding that a (one) unit can be more than one object takes time for learners to completely grasp.

Teach your students always to read fraction numbers like this: “three of four parts is three-fourths.” Allow plenty of time to practice, practice, and practice again reading fraction numbers accurately and concisely.

Example
What is the unit, or whole? This is the bottom number. (Five markers, or a set of five markers of two colors)

What fraction of the whole is red? This is the top number. (Two parts of five parts, or \(\frac{2}{5}\), are red.)

What fraction of the two-color whole is blue? (Three parts of five parts, or \(\frac{3}{5}\), are blue.)
When you teach basic fraction understanding, the instruction must be clear and simple. The focus must be on part-to-whole interpretation and area or region interpretation, using only \( \frac{1}{4} \), \( \frac{1}{2} \), and \( \frac{3}{4} \) (these three may be followed closely by \( \frac{1}{3} \) and \( \frac{2}{3} \)).

**Basic-Level Principle 2**

Introduce division in the early grades: present sharing and the concept of division simultaneously.

Fraction numbers can be presented in terms of sharing, and so they can also be used to explain the concept of division. An effective strategy for the acquisition of the concept of dividing is enhancing the child’s understanding of sharing. These two aspects of the mathematics curriculum, fraction numbers and division, can therefore be presented simultaneously and are appropriate to teach to young children.

**Example**

A pizza is shared equally among 4 children. How much of the pizza does each child receive?

We divide the pizza into 4 parts; each child gets 1 of the 4 parts, or \( \frac{1}{4} \) of the pizza.

In this beginning phase, your teaching emphasis must be on helping learners to understand the concepts embedded in Principle 1 and Principle 2, rather than on moving them quickly to symbol manipulations of fraction numbers.

**Basic-Level Principle 3**

Use equal sharing activities to introduce the concept of size in fraction numbers.

After learners have acquired a basic understanding of the small set of key fraction numbers they used with part-whole and area fraction models (\( \frac{1}{4} \), \( \frac{1}{3} \), \( \frac{1}{2} \), \( \frac{3}{4} \), \( \frac{2}{3} \), \( \frac{1}{3} \), \( \frac{1}{2} \)).
The next step is to practice ordering these fraction numbers from largest to smallest and smallest to largest. Be sure to restrict practice to the initial set until learners are confident in their knowledge of the value and magnitude of these five basic fraction numbers.

**Example**

Label index cards with the set of common fraction numbers \(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}\).

Students practice with the concept of order by shuffling the cards and arranging them from the largest to the smallest fraction number, then reshuffling and arranging them from the smallest to the largest.

---

### Decimal Number Strategies

In the usual mathematics curriculum sequence, decimal numbers are not introduced in the beginning grades. Even though learners don’t encounter decimal numbers until later in their formal education, experience has shown that many older students lack a conceptual understanding of decimal numbers. Since decimal numbers look like whole numbers, teachers often make the false assumption that they are easy to understand, and so may not emphasize the conceptual understanding of this mathematical form.

The fact of the matter is, young learners can grasp the concept of a decimal, and a solid foundation of decimal understanding must be built early. This foundation includes confidence with whole numbers, a clear understanding of place value, and a familiarity with how decimal numbers and fraction numbers are related. Decimal numbers are really fraction numbers that are written in different symbolic notation.

Historically, two different approaches to teaching decimal numbers have been used in formal instruction. One builds from a learner’s knowledge of place value in whole numbers. The other approach starts from a learner’s knowledge of fraction numbers. In the place-value approach, learners explore the concept by reading out each value in a whole and each value in a decimal number.

**Example**

Whole number 123:
- The place value of 3 is one-tenth the place value to the left.
- The place value of 2 is one-tenth the place value to the left and 10 times the place value to the right.
• The place value of 1 is 10 times the place value to the right and 100 times the place value two places to the right.

Decimal number 0.123:
• The place value of 1 is one-tenth the place value to the left and 10 times the place value to the right.
• The place value of 2 is one-tenth the place value to the left and 10 times the place value to the right.
• The place value of 3 is one-tenth the place value to the left and one-hundredth of the place value two places to the left.

Note: It is customary to include a zero before the decimal symbol on values less than one, e.g., 0.25 vs. .25. This indicates that there are no missing values to the left of the decimal symbol.

With the fraction approach, a clear understanding of how to read the decimal as a fraction is necessary. The common practice of reading decimal numbers by naming the digits makes it more difficult to think about decimal numbers as fractional numbers. The number 0.45 should not be read as “point four five” or “point forty-five,” but as “forty-five hundredths.” This practice of reading “point” in decimal numbers is like reading whole numbers one digit at a time; for example, reading 596 as “five nine six” rather than “five hundred ninety-six.” A learner must state the quantitative value of the digits when reading decimal numbers. Because of the common but inaccurate way of reading decimal numbers, many learners may need continuous practice in how to read a decimal number appropriately.

As they do with fraction numbers, teachers often tend to move too quickly to introduce the abstract decimal symbols. A deliberate sequence of instructional modes, from concrete to pictorial to abstract, must first be used to help students successfully grasp the concept of decimal numbers. A simple concrete and pictorial method for presenting the concept of a decimal number is the 10 x 10 grid (100 squares).

**Example**

The decimal number 0.50 could be shown on a 10 x 10 grid by shading in 5 columns or \( \frac{1}{2} \) of the grid. The shaded value could also be cut out and positioned on the other remaining part of the 10 x 10 grid to demonstrate that 0.50 and 0.50 combine to form 1.0 complete grid.
To develop basic understanding of decimal numbers in young learners, focus on only three decimal numbers: 0.25, 0.50, and 0.75. These values are the most common in young learners’ experiences and can be easily related to the respective fraction numbers they’ve learned. Later, too, they can be connected to the percentages they’ll learn.

Percent Strategies

In the traditional mathematics curriculum, percents are not usually introduced to beginning learners. Again, however, a foundation must be laid early to facilitate a more in-depth understanding later. At this first stage of development, relating the decimal numbers 0.10, 0.25, 0.50, and 0.75 to their counterpart percents, 10%, 25%, 50%, and 75%, is sufficient. These four percents will be the focus of the Basic-Level section.

After introducing the 10 x 10 grid for decimal numbers, it is natural to use the 10 x 10 grid to introduce percents. Some classroom teachers have found that explaining the % symbol as meaning “per 100”—two zeros with a slash mark—is helpful: 25% is also \( \frac{25}{100} \) or 0.25. Another helpful tip from classroom teachers is to tell students to think of percents as fraction numbers whose denominators (bottom numbers) are always 100: 75% is \( \frac{75}{100} \).

Developing the concept of percent using the instructional sequence of concrete to pictorial to abstract is again important. This instruction should build on the etymology of the word percent: per means “through or by” and cent means “hundred” in Latin, so 25% is “25 by the hundred,” or 25 parts per 100 parts. The 10 x 10 square grid is ideal to explore this basic meaning of percents; you can easily show how 10%, 25%, 50%, and 75% are all parts per 100. Later, use the 10 x 10 grid to explore equivalency: 25% is the same as \( \frac{25}{100} \) and \( \frac{1}{4} \) and 0.25. You can do the same for other basic percents (50%, 75%).

Informal Diagnostic Strategies

Young learners new to the concept of fraction numbers can misunderstand or make incorrect assumptions. If not addressed at this early stage, these conceptual errors can hinder students’ development of a sound fraction number sense. Teachers can use an informal diagnostic cycle to pinpoint and hopefully correct these misunderstandings before they lead to a more serious situation. First, the teacher detects an error pattern in a learner’s understanding of fraction number sense. The next stage is the most crucial: the diagnostic stage in which the teacher determines what error may be occurring. Then the teacher prescribes specific action, the learner works through the intervention, the teacher assesses the results, and the teacher evaluates whether the misunderstanding has been corrected. If the misunderstanding still exists, the cycle is repeated with new data.
This informal diagnostic strategy of detection, diagnosis, prescription, assessment, and evaluation can successfully help learners gain fraction number sense. Most misunderstandings at this stage of mathematical development are readily corrected if addressed soon. In this way young learners gain confidence in their mathematical abilities overall, and especially in their way around fraction numbers.

**Common Errors**

Some common misconceptions regarding fraction numbers arise when learners misinterpret pictorial representations or diagrams.

**Example**

What part of this circle is shaded?

Incorrect response: \( \frac{3}{5} \)

Incorrect reasoning: wrote fraction as \( \frac{\text{part}}{\text{part}} \) instead of \( \frac{\text{part}}{\text{whole}} \)

Correct response: \( \frac{3}{8} \)
Example

What part of the square is shaded?

Incorrect response: $\frac{1}{4}$ of the square is shaded.
Incorrect reasoning: did not consider that the parts are not equal in size (or area)

Correct response: The fraction of the square shaded is not $\frac{1}{4}$ but some other fraction.

Example

What parts are shaded?

Incorrect response: No fraction can be given.
Incorrect reasoning: Some learners do not believe this diagram shows equal fourths since the parts are not the same shape. Therefore, these learners believe the shaded parts cannot be expressed as a fraction.

Correct response: The parts are equal in area so $\frac{2}{4}$ or $\frac{1}{2}$ of the square is shaded.

Note: Equal areas do not need to be the same shape.
Example

What part is shaded? Then, locate the fraction on a number line.

Incorrect response: Some learners declare correctly that \( \frac{1}{2} \) is shaded. However, they cannot locate \( \frac{1}{2} \) on a number line.

Incorrect reasoning: Some learners hold fast to the belief that one-half is not a number; it is a part of a whole or in this case a square.

Correct response: \( \frac{1}{2} \)

These are just a few of the common misunderstandings we see among young learners; many others are possible. Therefore, as you teach fraction number sense, begin using the informal diagnostic cycle and interventions immediately.

Tips for Success

Here is a list of best practices for teaching beginning fraction number sense. These tips are not exhaustive; they are starting points for you to expand upon as appropriate in your learning environment.

- Explore how to share of a set of objects equally among a group.
- Have learners use words to explain the meaning of a fraction number.
- Keep the instructional mode sequence as the keystone focus: concrete to pictorial to abstract (touch it, see it, think it).
- Present sharing and the concept of division simultaneously.
- Focus on developmentally appropriate fraction numbers such as \( \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \).
- Model fraction situations with both area or region contexts as well as discrete objects.
- Don’t move ahead too quickly to the fraction symbols; students need to develop a solid conceptual understanding of fraction numbers first. This will
take time ... give your students that time.

- Help your students to develop an understanding of order, value, and magnitude as part of their basic fraction number sense.
- Begin teaching decimal numbers simultaneously with basic fraction numbers; show equivalency of 0.25 and $\frac{1}{4}$, 0.50 and $\frac{1}{2}$, and 0.75 and $\frac{3}{4}$.
- Keep instruction in percents limited at this stage of mathematical maturity.

**Protocols**

To successfully implement a fraction numbers program, teachers need to monitor learners’ progress during practice and to offer support when needed. The following protocols can be used as interventions, providing you with a basic framework for action. Each protocol identifies a specific topic and presents pre-assessment information, the appropriate development level, a sequence of actions, and an assessment strategy for evaluation. Also included are suggestions for adapting the protocol for special needs, gifted, and English-language learners.

The protocols are designed to take 20 to 30 minutes. The Launch section, which describes any preparation needed and instructions for teachers, may take 5 minutes; the Explore section, which describes what the students will do, 15 minutes; the Closure section 2 minutes and the Assessment and Evaluation sections together another 5 minutes. Of course, the protocols are quite flexible; you may modify them as needed for the most effective use in your particular learning environment. The time frame of each protocol can easily be adapted to any learning environment as well.

The protocols in this Basic-Level section address simple fraction numbers and their corresponding decimal numbers and percents, though with a focus on fraction numbers. Some of the activities are appropriate for two to four learners and some may involve the entire class. Many of the protocols can also be completed by an individual learner. The protocols require only minimal materials such as counters, number cubes, pattern blocks, and circular, square, and rectangular pieces. You may need to create some of these, but they are easy to make. We have provided reproducible forms for some instructional aids in the appendix. You can make number cubes by taping number values over the dots on dice. Some protocols require a set of pattern blocks; these are usually readily available within the current mathematics curriculum.
The matrix below lists the protocols for basic-level learners.

### Basic-Level Protocols

<table>
<thead>
<tr>
<th>Title</th>
<th>Topic</th>
<th>Instructional Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make a Whole</td>
<td>forming the unit or whole</td>
<td>individual, small group</td>
</tr>
<tr>
<td>Make a Half</td>
<td>fraction numbers equivalent to $\frac{1}{2}$</td>
<td>individual, small group</td>
</tr>
<tr>
<td>In-Between Numbers</td>
<td>magnitude of fraction numbers on a number line</td>
<td>whole group, small group</td>
</tr>
<tr>
<td>Comparing Fraction Numbers</td>
<td>fraction numbers less than, equal to, or greater than</td>
<td>small group</td>
</tr>
<tr>
<td>$&lt; = &gt;$ (Less Than, Equal To, Greater Than)</td>
<td>fraction numbers less than, equal to, or greater than $\frac{1}{2}$</td>
<td>small group</td>
</tr>
<tr>
<td>Fractions Using Interlocking Circles</td>
<td>parts, wholes, and complements</td>
<td>whole group, individual</td>
</tr>
<tr>
<td>Fraction Block Covers</td>
<td>basic fraction numbers $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$</td>
<td>whole group, individual</td>
</tr>
<tr>
<td>Block Trades</td>
<td>exchanging fraction values</td>
<td>small group</td>
</tr>
<tr>
<td>Close Is Just Fine</td>
<td>measurement with fraction numbers $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$</td>
<td>small group, individual</td>
</tr>
<tr>
<td>Grid Lock</td>
<td>decimal numbers</td>
<td>individual</td>
</tr>
<tr>
<td>Shade the Percents</td>
<td>percents</td>
<td>individual</td>
</tr>
<tr>
<td>Match Them All</td>
<td>relationship of fraction numbers, decimal numbers, and percents</td>
<td>small group</td>
</tr>
</tbody>
</table>
Make a Whole

**Topic:** Forming the unit or whole

**Level:** Basic

**Instructional Mode:** Individual, small group (two learners)

**Time:** 20–30 minutes

**Pre-assessment**
Check knowledge that a whole can be divided into equal parts.

**Materials**
- Fraction Circles, one set for an individual or for two learners
- Use only \( \frac{1}{2} \), \( \frac{1}{3} \), \( \frac{1}{4} \), and \( \frac{1}{6} \) circles. Later, you may use more fraction sets.

(Reproducible circles are provided in the appendix.)

**Launch**
- Introduce the Fraction Circles.
- Let the learners explore with the Fraction Circles.
- Walk around and watch what the learners are doing while they explore.
- Ask the learners questions about the Fraction Circles.

**Explore**
- Ask the learners to put all the same colored Fraction Circles together to form a circle (or the whole). Then ask a series of questions:
  - How many of the \( \frac{1}{4} \) circles (or pieces) does it take to create a whole (unit)?
  - How many of the \( \frac{1}{3} \) circles does it take to create a whole?
  - How many of the \( \frac{1}{2} \) circles does it take to create a whole?
  - How many of the \( \frac{1}{6} \) circles does it take to create a whole?
- Continue until all circles have been put together.

**Closure**
- Ask:
  - How many fourths make a whole? (\( \frac{4}{4} \))
  - How many thirds make a whole? (\( \frac{3}{3} \))
  - How many halves make a whole? (\( \frac{2}{2} \))
<table>
<thead>
<tr>
<th>Assessment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask learners if they see a pattern.</td>
</tr>
<tr>
<td>Let them come up with their own wording, such as, &quot;If the top number and</td>
</tr>
<tr>
<td>bottom number are the same, the fraction equals one whole.&quot;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success will be shown when the learner recognizes when a fraction is one</td>
</tr>
<tr>
<td>whole, or one.</td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
- Use a diagram or circle to explain what it means that something is a whole or a unit.
- Review the words *fourths, thirds, halves, and sixths*.

**Special Needs**
- Use only the $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ Fraction Circles.

**Gifted**
- Ask learners how many halves, thirds, fourths, and sixths it takes to make two units and three units.
Make a Half

**Topic:** Fraction numbers equivalent to a half
**Level:** Basic
**Instructional Mode:** Individual, small group (two learners)
**Time:** 20–30 minutes

**Pre-assessment**
Check for understanding of the concept of $\frac{1}{2}$.

**Materials**
- Fraction Circles, in the beginning, use only $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$ circles.
  (Reproducible circles are provided in the appendix.)

**Launch**
- Have learners explore with the Fraction Circles.
- Determine whether learners can find the $\frac{1}{2}$ piece.
- Ask learners to create the $\frac{1}{2}$ shape using only one color of the Fraction Circles.

**Explore**
- Have learners tell what they found.
- For example, you need two of the $\frac{1}{4}$ pieces to make the $\frac{1}{2}$ shape, or $\frac{2}{4}$ is the same as $\frac{1}{2}$.
- Have learners record other examples that form the $\frac{1}{2}$ shape, e.g., $\frac{4}{8}$, $\frac{6}{12}$, $\frac{5}{10}$, $\frac{3}{6}$.

**Closure**
- Have learners create a “rule” to be able to recognize when a fraction is equal (equivalent) to $\frac{1}{2}$.
- For example, “The top number is half of the bottom number” or “The bottom number is twice the top number” or “The bottom number is always an even number.”
Assessment

Show several fraction numbers and ask if they are equal, or equivalent, to the $\frac{1}{2}$ shape.

For example, show $\frac{3}{4}, \frac{4}{8}, \frac{6}{12}$, and $\frac{4}{8}$.

<table>
<thead>
<tr>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success will be shown when a learner can look at any proper fraction and determine if it is equal or equivalent to $\frac{1}{2}$.</td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL

• Focus more on the manipulatives and hands-on materials while emphasizing the appropriate vocabulary.

Special Needs

• Focus only on fraction numbers that have even number denominators.

Gifted

• Give more attention to fraction numbers with odd number denominators.
• Ask learners to articulate why a fraction is not equal to $\frac{1}{2}$. 
In-Between Numbers

**Topic:** Magnitude of fraction numbers on a number line  
**Level:** Basic

**Instructional Mode:** Whole group, small group (three or four learners)  
**Time:** 20–30 minutes

**Pre-assessment**
- Check for understanding of a number line.
- Review the meaning of the phrases *greater than* and *less than*.

**Materials**
- Fraction Number Cards and Fraction Circles; use $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{8}$ sets  
  *(Reproducible cards and circles are provided in the appendix.)*
- Teacher-constructed enlarged number line starting at 0 and ending at 1 with no hash marks

**Launch**
- Show how you can place the Fraction Number Card on the number line.
- Demonstrate how to say the fraction out loud and then place the card on the approximate location on the number line.

**Explore**
- Each learner should explore by placing the different Fraction Number Cards on the number line.
- If learners are completing this activity as a small group, the other learners should discuss each learner’s choice of location for each Fraction Number Card.

**Closure**
- Review what it means to be approximately in the correct location on the number line.
- Place the $\frac{2}{3}$ Fraction Number Card to the left of the $\frac{3}{4}$ Fraction Number Card on the number line.
- Use the Fraction Circles to show that the $\frac{3}{4}$ Fraction Number Card is larger than the $\frac{2}{3}$ Fraction Number Card.
### Assessment

| Allow the learner to select a Fraction Number Card. |
| Have the learner place this card on its approximate location on the number line. |

### Evaluation

Success will be shown when learners can accurately place proper fraction numbers on a number line.

---

#### Adaptations for Various Learners

**ELL**
- Show the meaning of *less than* and *greater than* on a number line with whole number values.
- Review the words *halves, thirds, fourths*, and *eighths*.

**Special Needs**
- Use the Fraction Number Cards for \(\frac{1}{4}\)s, \(\frac{1}{3}\)s, and \(\frac{1}{2}\)s.
- Construct the number line with some benchmarks to help learners find the approximate location for the Fraction Number Cards.

**Gifted**
- Use as many Fraction Number Cards as appropriate.
- Discuss the meaning of in-between numbers for whole numbers and for fraction numbers.
Comparing Fraction Numbers

Topic: Fraction numbers less than, equal to, or greater than
Instructional Mode: Small group (two learners)
Level: Basic
Time: 20–30 minutes

Pre-assessment
• Check understanding of the concepts less than, equal to, and greater than.

Materials
• Fraction Circles and Fraction Number Cards
  (Reproducible cards and circles are provided in the appendix.)

Launch
• The two players will need a set of Fraction Circles and a set of Fraction Number Cards.
• The chosen set of Fraction Circles and Fraction Number Cards depends on the players’ level of concept development.
• Shuffle each set of Fraction Circles and Fraction Number Cards.
• Place the cards face down into two decks, Fraction Circles and Fraction Number Cards.

Explore
• Each player draws a card from either the Fraction Circles or Fraction Number Cards. Players can draw from the same set.
• The player with the larger fraction takes both cards for that round of play.
• If the cards are equal, each player draws again. Now the player with larger fraction takes all four cards.
• Continue playing until all the cards are taken by the players.
• The player with the most cards could be declared the winner.
• Play the game again but this time the player with the smaller fraction takes both cards for that round of play.

Closure
• Remind the learners that fraction numbers can be larger and smaller when they are compared.
• Select two cards and ask which is smaller.
### Assessment

<table>
<thead>
<tr>
<th>Task</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Select any two cards. Ask the learners which card is larger.</td>
<td>Success will be shown when the learner can compare any two proper fraction numbers and declare which one is larger.</td>
</tr>
<tr>
<td>Select another two cards. Ask the learners which card is smaller.</td>
<td>Success will be shown when the learner can compare any two proper fraction numbers and declare which one is smaller.</td>
</tr>
<tr>
<td>Compare the selected cards to $\frac{1}{2}$ and 1 as benchmarks.</td>
<td></td>
</tr>
</tbody>
</table>

### Adaptations for Various Learners

**ELL**
- Review the meaning of the words *largest* and *smallest*.
- Play a round with the learner to review the sequence of how to play the game.
- Demonstrate what to do when the two drawn cards are equal.

**Special Needs**
- Use only the Fraction Circles that are developmentally appropriate for the learner.
- Have the players only compare the size of the Fraction Circles to determine the winner of the round.

**Gifted**
- Use only the Fraction Number Cards that are developmentally appropriate for the learner.
- Have the learner draw at random three cards.
- Have the learner place these three cards in order from smallest to largest or from largest to smallest.
- Repeat the drawing at random until many Fraction Number Cards have been selected.
< = > (Less Than, Equal To, Greater Than)

**Topic:** Fraction numbers less than, equal to, or greater than $\frac{1}{2}$  
**Level:** Basic  
**Instructional Mode:** Small group (two or three learners)  
**Time:** 20–30 minutes

### Pre-assessment
- Check for understanding of the concept of one-half ($\frac{1}{2}$).
- Check for understanding of other ways of representing $\frac{1}{2}$.

### Materials
- Fraction Circles and Fraction Number Cards  
  *(Reproducible cards and circles are provided in the appendix.)*
- three index cards labeled separately with the symbols <, =, and >

### Launch
- Review the meaning of <, =, and >.
- Select the Fraction Number Cards appropriate for the development level of the small group.
- Match the Fraction Number Cards with their corresponding Fraction Circles.
- Place the index cards with the symbols <, =, and > in a row.
- Explain that the Fraction Number Cards will be placed face down in a deck.
- Each learner will draw a Fraction Number Card from the deck.
- The learner must decide how the fraction compares to $\frac{1}{2}$. Is it less than, equal to, or greater than? The learner places the card under the correct symbol. For example, if $\frac{1}{3}$ were drawn, it would go under the < symbol, since it is less than $\frac{1}{2}$.

### Explore
- Start the activity with the learners drawing the Fraction Number Cards.
- Have the learners place their Fraction Number Card under the correct symbol relative to $\frac{1}{2}$.
- If learners cannot determine the size of the Fraction Number Card relative to $\frac{1}{2}$, they can use the Fraction Circles to help them find the answer.

### Closure
- Review with the learners how they determined where to put the cards.
- Use the Fraction Circles to demonstrate how to compare the fraction magnitude to $\frac{1}{2}$. 
### Assessment

Select a Fraction Number Card.

Ask the learner if it is greater than, equal to, or less than the fraction $\frac{1}{2}$.

Name a proper fraction and ask the learner if it is greater than, equal to, or less than the fraction $\frac{1}{2}$.

### Evaluation

Success will be shown when learners are confident they can determine the relative size of any proper fraction to the benchmark of $\frac{1}{2}$.

### Adaptations for Various Learners

**ELL**
- Review the meaning of the words *greater than*, *less than*, and *equal to*.
- Match the symbols $<$, $=$, and $>$ with the words.

**Special Needs**
- Choose Fraction Number Cards that are developmentally appropriate.
- Use the corresponding Fraction Circles to check for greater than, equal to, and less than for every drawn Fraction Number Card.
- Make a matrix with the index cards and Fraction Number Cards to ensure proper alignment.

**Gifted**
- Enlarge the set of Fraction Number Cards to include all appropriate fraction numbers for skill level.
- Replace the benchmark $\frac{1}{2}$ with the benchmark $\frac{1}{4}$. Now complete the activity.
- Replace the benchmark $\frac{1}{4}$ with the benchmark $\frac{2}{3}$. Now complete the activity.
Fractions Using Interlocking Circles

Topic: Parts, wholes, and complements
Level: Basic
Instructional Mode: Whole group, individual
Time: 20–30 minutes

Pre-assessment
- Check for basic understanding of parts and the whole.

Materials
- Interlocking Circles of two different colors, for example, red and blue
  *(Reproducible circles are provided in the appendix.)*

Launch
- Demonstrate how to cut out the two circles and how to cut each along its radius.
- Put the two circles together so that they interlock.
- Demonstrate how to rotate the circles to form different fraction numbers.
- Point out that the two sides of the Interlocking Circles form complements.

Explore
- Allow learners to explore with Interlocking Circles. They could show their partners the different fraction numbers that they formed.
- Ask a set of questions such as:
  - How can you show $\frac{1}{4}$ red?
  - How can you show $\frac{1}{2}$ red?
  - How can you show $\frac{1}{3}$ blue? What would the red fraction be now?
  - How can you show a fraction larger than $\frac{1}{2}$?
  - How can you show a fraction smaller than $\frac{3}{4}$?

Closure
- Ask the question: If there is $\frac{1}{4}$ red on one side of the Interlocking Circles, how much red is on the other side?
- Repeat the same type of question with other fraction numbers.
Assessment

Form the learners into pairs.

Have the learners ask each other questions, e.g., 
How do you show the fraction $\frac{2}{3}$?

Check for accuracy in answering using the Interlocking Circles.

<table>
<thead>
<tr>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success will be shown when learners demonstrate a consistently accurate visual representation of various fraction numbers.</td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL

• Review the meaning of the words parts and whole.

• Explain what you mean by interlocking.

Special Needs

• Construct the Interlocking Circles beforehand for the learner.

• Demonstrate how to place the circles together to make them rotate smoothly.

• Choose colors that visually challenged learners would be able to see clearly.

Gifted

• Use the Interlocking Circles to gather information or data.

• For example, if red represented strawberries and blue represented blueberries, have the learner ask 10 or 12 other learners which fruit they like better.

• Use the Interlocking Circles to show the favored fruit results for the 10 or 12 learners.

• This could be an introduction to representing data using circle graphs.
Fraction Block Covers

**Topic:** Basic fraction numbers ($\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$)  
**Level:** Basic  
**Instructional Mode:** Whole group, individual  
**Time:** 20–30 minutes

**Pre-assessment**
- Check for understanding of what *cover* indicates.

**Materials**
- Pattern blocks: hexagon, trapezoid, rhombus, and triangle

**Launch**
This activity can be used for review of, or practice with, the basic fraction numbers $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$.
- Show how to cover the trapezoid with three triangles.
- Review the idea that the trapezoid is the whole, or unit.
- Count the number of parts. (3)
- Review concept that each part is 1 of 3, or $\frac{1}{3}$.

**Explore**
Have learners
- Use the trapezoid blocks to cover the hexagon.
- Show the part and whole with a trapezoid and hexagon.
- Write the part/whole: $\left(\frac{1}{2}\right)$.
- Use the rhombus blocks to cover hexagon.
- Show the parts and whole with the rhombus and hexagon. ($\frac{1}{3}$, $\frac{2}{3}$)
- Use the rhombus and a triangle to cover the trapezoid.
- Show the part and whole with the rhombus and trapezoid. ($\frac{2}{3}$)
- Show the part and whole with the triangle and trapezoid. ($\frac{1}{3}$)

**Closure**
- Review by demonstrating the fraction numbers $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$ with the hexagon, trapezoid, and rhombus.
Assessment
Take the hexagon and ask the learner to demonstrate $\frac{1}{3}$, using the rhombus.

Evaluation
Success is the ability to demonstrate the part and whole using one rhombus and one hexagon.

Adaptations for Various Learners

ELL
- Demonstrate with both trapezoid and hexagon what cover means.
- Demonstrate how to write part and whole symbolically.

Special Needs
- Demonstrate the action of covering using the pattern blocks.
- Demonstrate how to write part and whole symbolically.
- Use only trapezoid and triangle and show only the fraction numbers $\frac{1}{2}$ and $\frac{2}{3}$.

Gifted
Have learners
- use the triangles to cover the hexagon.
- demonstrate $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$.
- write part and whole symbolically with 6s.
- write $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{6}{6}$ in another way, if possible. ($\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$)
Block Trades

**Topic:** Exchanging fraction values

**Instructional Mode:** Small group (three to five learners)

**Level:** Basic

**Time:** 20–30 minutes

**Pre-assessment**
- Learners should be familiar with pattern blocks, particularly the hexagon, trapezoid, rhombus, and triangle.

**Materials**
- pattern blocks: hexagons, trapezoids, rhombi, and triangles
  (You can use the term fraction blocks instead of pattern blocks when introducing this activity.)
- a number cube, faces labeled with the numerals 1, 2, 3, 4, 5, 6

**Launch**
Have students get in small groups of three to five learners. The object is to accumulate five hexagons. Explain the directions of this activity and/or demonstrate as follows:
- Select a learner to be the banker. This participant gives out pattern blocks to the players as required.
- Each learner rolls the number cube in turn. The banker gives the learner the number of triangles indicated on the face of the number cube.
- The learner then trades for equivalent larger blocks.
- Each learner should have the fewest number of blocks possible at all times. For example, if a learner rolls a 5, he or she receives five triangles from the banker and then trades them for one trapezoid and one rhombus.

**Explore**
Have learners
- do the Block Trades activity for four or five rounds.
- describe their strategies for covering the pattern blocks with triangles.
- search for a strategy of making the most efficient exchanges in order to accumulate five hexagons.

**Closure**
- Check for accuracy in exchanging the appropriate pattern blocks.
**Assessment**

Give a learner five triangles, one rhombus, and one trapezoid.

Ask what would be the most efficient block exchanges.

**Evaluation**

Success is showing that a trapezoid could be made with three of the five triangles and a rhombus can be made from two triangles. Now the two trapezoids can form a hexagon.

Result: one hexagon and two rhombi

---

**Adaptations for Various Learners**

**ELL**

- Using the pattern blocks, review the definitions of rhombus, triangle, hexagon, and trapezoid.
- Have learners record the number rolled with the number cube.

**Special Needs**

- When introducing the Block Trades activity, use only the trapezoid, rhombus, and triangle.
- The object of the activity would now be to form five trapezoids from the rhombi and triangles.

**Gifted**

- Have the learners write out or diagram a strategy for trading six triangles, five triangles, and four triangles.
- Start with the hexagon and use the number cube to remove triangles. Find a pattern with the removal activity.
Close Is Just Fine

**Topic:** Measurement with fraction numbers  
**Level:** Basic

**Instructional Mode:** Two learners, individual  
**Time:** 20–30 minutes

### Pre-assessment
- Learners should know how to measure with a ruler by using the zero or left edge of the device.

### Materials
- specific ruler (could be 1 inch long) marked with only 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1
- another specific ruler marked with only $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \text{and } \frac{8}{4}$
- 10–20 objects between $\frac{1}{4}$ inch and 2 inches long
- a recording sheet with a place to draw the object and record the close measure

*(A reproducible form is provided in the appendix.)*

### Launch
- Using the smaller ruler, explain the procedure for measuring an object.
  - Place the left edge (or zero) at the left edge of the object.
  - Look at the right edge of the object and find the closest mark.
  - Assist in reading the close value.
- Using the larger ruler, explain the procedure for measuring the object.
  - Assist in reading the close value, e.g., $\frac{5}{4}$.

### Explore
Have learners
- choose a variety of 8 to 10 objects which are between $\frac{1}{4}$ inches and 2 inches long from the teacher-collected set of objects.
- start with the smaller ruler and some objects between $\frac{1}{4}$ inch and 1 inch long.
- measure each object and record the close value on the recording sheet. An object measuring between $\frac{1}{2}$ and $\frac{3}{4}$ can have a close value of either $\frac{1}{2}$ or $\frac{3}{4}$. Do not be concerned about absolute precision at this time.
• move on to the larger ruler.
• measure the objects (some less than 1 inch and some more than 1 inch but less than 2 inches).
• record the information regarding the close value on the recording sheet.

Closure
• Review the procedure for measuring objects with a marked ruler.
• Show what close means in measuring on special rulers.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>See if learners can use the smaller ruler to measure an object between $\frac{1}{4}$ inch and 1 inch long.</td>
<td>Learners have successfully grasped the concept if they declare the close measurement value accurately.</td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
• Provide extra practice time for measuring, and emphasize the meaning of close.
• Use only the smaller ruler.

Special Needs
• Provide extra practice time for lining up the left edge of the ruler and the object.
• Practice reading the close values together with learners.
• Use only the smaller ruler.

Gifted
• Enlarge one of the specific rulers to 3 or 4 inches.
• Have learners write the fraction numbers larger than 1 as mixed numbers, e.g., $\frac{5}{4}$ as $1 \frac{1}{4}$.
• Have learners construct a 12-inch ruler with the approximate markings in $\frac{1}{4}$ and maybe $\frac{1}{8}$ inches.
Grid Lock

**Topic:** Basic decimal numbers (0.25, 0.50, and 0.75)  
**Level:** Basic  
**Instructional Mode:** Individual  
**Time:** 20–30 minutes

**Pre-assessment**
- Check that learners can say and write the basic decimal numbers.

**Materials**
- a set of 10 x 10 square grids  
  *(A reproducible form is provided in the appendix.)*  
- a set of instruction cards (made from index cards) with the following information:
  - show 0.25
  - show 0.50
  - show 0.75
  - 0.25 and 0.75 together make what value?
  - 0.25 and 0.50 together make what value?

**Launch**
- Prepare a 10 x 10 grid with 25 squares shaded (not necessary that the squares are all connected, but helpful if they are).
- Explain the 10 x 10 grid to students.
  - number of rows (10)
  - number of columns (10)
  - number of total squares (100)
- Show the prepared grid with 25 squares out of 100 squares shaded, and explain that this represents \( \frac{25}{100} \) or 0.25.

**Explore**
Have learners
- pick an instruction card and shade the value on a 10 x 10 grid. Learners could also cut out the shaded part.
- pick another card and shade the value on the 10 x 10 grid.
- continue picking cards until all have been used.
Closure

- Have learners make up a new instruction card, writing down both the instruction and the answer. This value must be able to be shown on the 10 x 10 grid.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shuffle the cards.</td>
<td>Success is showing the card value accurately on the grid. Writing the correct decimal expression is further evidence of success.</td>
</tr>
<tr>
<td>Draw a card. Do not show the card to the learner.</td>
<td></td>
</tr>
<tr>
<td>Ask the learner to show the value on the grid.</td>
<td></td>
</tr>
<tr>
<td>Probe the learner to write the value as a decimal number.</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
- Explain what is usually meant by the word *show*.
- Explain what is meant by the word *value*.

Special Needs
- Have two learners work together.
- One learner picks the instruction card and explains what it says. The other learner shows the value on the 10 x 10 grid.
- Modify the 10 x 10 grid by adding labels 10, 20, 30, etc. to the columns.
- Have learners practice with the one-value decimal cards before introducing the show-together value cards.

Gifted
- Challenge learners to illustrate 0.25 in other ways besides shading two complete columns and five squares.
- Challenge learners to find the value of 0.75 + 0.50, 0.25 + 0.50 + 0.75, and -0.25 + 0.50. Have learners show these results on the 10 x 10 grid.
Shade the Percents

**Topic:** Percents 10%, 25%, 50%, and 75%  
**Level:** Basic  
**Instructional Mode:** Individual  
**Time:** 20–30 minutes

**Pre-assessment**
- The learner must be able to say and write basic percents.

**Materials**
- a set of 10 x 10 grids  
  
  *(A reproducible form is provided in the appendix.)*  
- a set of seven index cards, each with one of the following instructions:  
  - show 10%  
  - show 25%  
  - show 50%  
  - show 75%  
  - show a percent value less than 50%  
  - show a percent value more than 75%  
  - show a percent value between 25% and 75%

**Launch**
- Explain the 10 x 10 grid for percents: number of rows (10), number of columns (10), and total number of squares (100).  
- Demonstrate by shading 10 squares.  
- Explain why this is 10% or 10 parts (squares) of 100 parts (squares) or \( \frac{10}{100} \) or 0.10.

**Explore**
Have learners
- pick an instruction card and shade the given value on a 10 x 10 grid. They could also cut out the shaded part.  
- pick another card and shade the value on a new 10 x 10 grid.  
- continue picking cards until all the cards have been used.

**Closure**
- Have the learner make a new instruction card, writing down the correct answer as well. The value must be able to be shown on the 10 x 10 grid.
Assessment | Evaluation
--- | ---
Shuffle the cards. | Success is shading the grid accurately. Writing the correct percent expression is further evidence of success.
Draw a card. Do not show the card to the learner. |  
Ask the learner to show this value on the 10 x 10 grid. |  
Probe the learner to write the answer as a percent. |  
Success is shading the grid accurately. Writing the correct percent expression is further evidence of success.

Adaptations for Various Learners

ELL
- Explain what is usually meant by the word show.
- Explain the basic meaning of the symbol % (parts of a hundred).

Special Needs
- Allow two learners to work together.
- One learner would pick the instruction card and explain what it says. The other learner would show the value on the grid.
- Modify the 10 x 10 grid by labeling the columns with 10, 20, 30, 40, etc.
- Practice with just the one-value instruction cards before introducing the second set of more complex instruction cards.

Gifted
- Challenge learners to find multiple ways to show 25% on the 10 x 10 grid.
- Challenge learners to find $33\frac{1}{3}\%$ and $66\frac{2}{3}\%$ on the 10 x 10 grid.
Match Them All

**Topic:** Relationship of fraction numbers, decimal numbers, and percents  
**Level:** Basic  
**Instructional Mode:** Small group (two to four learners)  
**Time:** 20–30 minutes

**Pre-assessment**
- Check that learners can recognize the fraction numbers $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$; the decimal numbers 0.25, 0.33, 0.50, 0.67, and 0.75; and the percents 25%, 33%, 50%, 67%, and 75%.

**Materials**
- three number cubes with specially labeled faces:
  - a fraction cube with faces labeled $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and BONUS
  - a decimal cube with faces labeled 0.25, 0.33, 0.50, 0.67, 0.75, and BONUS
  - a percent cube with faces labeled 25%, 33%, 50%, 67%, 75%, and BONUS
- a felt pad or mat to reduce the noise generated by rolling the number cubes
- recording score card

*(A reproducible form is provided in the appendix.)*

**Launch**
- Review the values on each face of the fraction cube, decimal cube, and percent cube.
- Explain the rules of the activity.
  - Take turns rolling the number cubes.
  - The first learner rolls all three cubes.
    - If no matches occur, zero points
    - If two values match, for example 25% and 0.25, 3 points
    - If three values match, for example $\frac{1}{2}$, 0.50, and 50%, 6 points
  - BONUS can be used to form a match. However, the learner must declare the match value. If a learner rolls the values $\frac{2}{3}$, 25% and BONUS, for instance, he or she may declare the BONUS as 67% or 0.67 or 0.25 or $\frac{1}{4}$ to create a match and earn 3 points. Any of these choices would be correct. In addition, two BONUS faces showing can form a match, resulting in 6 points.
  - Continue the activity until each learner reaches at least 100 points.
Explore
• Allow learners to explore the Match Them All activity for 20 to 30 minutes.
• Check that learners are recording the scores accurately.
• Check that learners are accurately declaring the value of the BONUS.

Closure
• Review the relationship of fraction numbers, decimal numbers, and percents by giving examples that are related, e.g. $\frac{3}{4}$, 0.75, and 75%.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll the three number cubes.</td>
<td>Success occurs if the learner can</td>
</tr>
<tr>
<td>Have the learner describe the match or matches that occurred.</td>
<td>• match the fraction number, decimal number, and percent correctly.</td>
</tr>
<tr>
<td>Roll again until a BONUS appears face up. Ask the learner to state a possible match.</td>
<td>• articulate the values that match using the BONUS feature.</td>
</tr>
</tbody>
</table>

Assessment Evaluation

<table>
<thead>
<tr>
<th>0.25</th>
<th>0.33</th>
<th>0.50</th>
<th>0.67</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{4})</td>
<td>25%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{1}{3})</td>
<td></td>
<td></td>
<td></td>
<td>33%</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td></td>
<td></td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>(\frac{2}{3})</td>
<td></td>
<td></td>
<td></td>
<td>67%</td>
</tr>
<tr>
<td>(\frac{3}{4})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
• Review the meaning of the symbols of fraction numbers, decimal numbers, and percents.
• Explain the meaning of the word bonus as a special deal or feature.

Special Needs
• Make a chart for learners to use as a reference during the activity. Fraction numbers and decimal numbers are matched with the appropriate percent. (A reproducible form is provided in the appendix.)

Gifted
• Have learners explore the quickest route to 100 points. This may not be getting three matches. The BONUS feature can be a factor in reaching 100 points at a quick pace.
  - For example, roll #1 yields \(\frac{1}{2}\), 50%, and 0.50 (6 points); roll #2 yields \(\frac{1}{3}\), 0.33, 0.75 (3 points) for a total of 9 points.
  - For example, roll #3 produces \(\frac{1}{2}\), BONUS, BONUS (6 points); roll #4 yields \(\frac{1}{3}\), BONUS, BONUS (6 points) for a total of 12 points.
Developing-Level Fraction Numbers

59  Fraction Number Strategies
63  Decimal Number Strategies
69  Percent Strategies
72  Informal Diagnostic Strategies
73  Common Errors
75  Tips for Success

76  Protocols
78  Marker Cover Up
80  Equivalent Match
82  Make Equivalent Fractions
84  Numerator and Denominator
86  Decimal Number Cover
88  Checking 0.50
90  Decimal Number Comparison
92  Decimal Names
94  Money Bags
96  Percent Number Line
98  Percents with Interlocking Circles
100 Grid Percent
Developing-Level Fraction Numbers

The Basic-Level section of this guide contains strategies to give learners a solid foundation in simple fraction numbers, decimal numbers, and percents. This section, aimed at developing-level learners, builds upon that foundation to include complex fraction numbers, more aspects of decimal numbers, and additional values of percents. The fundamental concepts of fraction numbers, decimal numbers, and percents are reviewed and reinforced, as are the three principles explored in the Basic-Level section.

Ensure Basic-Level Understanding First
If you are using this guide in an intervention setting, make sure your students develop a sound fraction number sense by reviewing concepts at the basic level before working in the developing or advanced level. A beginning fraction number sense is crucial regardless of the age or grade level of your students.

Basic-Level Principle 1
Build upon knowledge students already possess: introduce fraction numbers by having students explore how to share a set of objects equally among a group.

Basic-Level Principle 2
Introduce division in the early grades: present sharing and the concept of division simultaneously.

Basic-Level Principle 3
Use equal sharing activities to introduce the concept of size in fraction numbers.

These Basic principles lead to the three principles that underlie the Developing-Level teaching strategies:
Developing-Level Principle 1
Emphasize that fractions are numbers, and they have magnitude.

Developing-Level Principle 2
Explore equal sharing activities to develop learners’ understanding of fractional equivalence.

Developing-Level Principle 3
Use a number line to represent fractions and decimal numbers; this builds appropriate understanding.

At the developing level, emphasize the relationship of fraction numbers to decimal numbers and percents. These three number values are very closely aligned, so it’s best to explore them as a coherent set of numbers, not as separate entities. This section also stresses the importance of exploring decimal and fraction numbers in terms of measurement. Students need to understand fraction numbers to use our customary measurement system based on the English inch, foot, and yard. For example, the fractions $\frac{3}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ are commonly used lengths. And when using the metric measurement system in scientific or other contexts, learners need a firm understanding of decimal numbers.

Some learners become overwhelmed when faced with fraction numbers in any but very simple situations. This guide focuses on two instructional aids that make explaining and demonstrating fractions easy. The first is the number line, which you can use to introduce proper and improper fraction numbers, decimal numbers, and percents. The number line is easy to construct, and it reinforces Developing-Level Principle 1: fractions are numbers, and they have magnitude. The other aid is a 10 x 10 grid of 100 squares, which is good for demonstrating the interconnectedness of fraction numbers, decimal numbers, and percents. These two instructional aids are appropriate for learners at any stage of understanding fractions. They are also helpful in exploring equivalent fractions, a key concept at the developing level.

Like the Basic-Level section, the Developing-Level section is organized into six areas: fraction number strategies, decimal number strategies, percent strategies, informal diagnostic strategies, common errors, and tips for success. These strategies are not exhaustive; they are sample strategies that have proven successful for some Developing-Level learners. The ten tips for success offer a quick reference of best practices for further developing students’ understanding of fraction numbers.

Twelve protocols appropriate for developing-level study follow the discussion of general strategies. Four focus on fraction numbers, four on decimal numbers,
Developing-Level Fraction Numbers

one on the relationship of fractions and decimals leading to percents, and three on percents. The protocols are not complete lesson plans or curriculum units; instead, consider them as templates with a specific focus that you can modify as needed to suit your day’s lesson plan. While designed primarily for small- and whole-group instruction, the protocols can be used for individual tutoring by an educational aide or parent.

Fraction Number Strategies

For learners to acquire expanded knowledge of fraction numbers, they must understand the concept of fractions. Therefore, even at this developing stage, teachers need to pay careful attention to misunderstandings, incorrect reasoning, and consistent error patterns among their students. By building on the strategies presented in the Basic-Level section, you can help learners acquire both confidence and skill in working with fraction numbers.

A key principle for building on learners’ foundational fraction number sense is the following:

**Developing-Level Principle 1**

Emphasize that fractions are numbers, and they have magnitude.

For some learners, fractions are mysterious. But if you consistently show them that fractions are numbers just like whole numbers, you can dispel the mystery. One way of explaining their number quality is to demonstrate that fraction numbers are the numbers between the whole numbers. You can show this on the number line. Also, you can show that between any two fraction numbers there are more fraction numbers.

**Example**

Between $\frac{3}{8}$ and $\frac{7}{8}$, you will find $\frac{4}{8}$, $\frac{5}{8}$, and $\frac{3}{4}$, to name just some that come between them.

Another concept to introduce at this level is equivalency. Fraction numbers can be expressed in many ways. For example, the fraction number $\frac{6}{8}$ can also be expressed as $\frac{3}{4}$. In other words, the fraction $\frac{6}{8}$ and the fraction $\frac{3}{4}$ are equivalent fractions: they have the same value and magnitude. Even though the numerals 3 and 4 are smaller than 6 and 8, the fraction number $\frac{3}{4}$ is not smaller than $\frac{6}{8}$, as some learners believe. Therefore, remember the second
important principle for teaching at this stage of fraction development:

### Developing-Level Principle 2

Explore equal sharing activities to develop learners’ understanding of fractional equivalence.

In conjunction with exploring equivalent fraction numbers, learners should develop the skill of renaming fraction numbers. The fraction \( \frac{1}{2} \) can be renamed \( \frac{2}{4} \) when the numerator and denominator are both multiplied by 2. This multiplication does not change the magnitude of the fraction number (since \( \frac{2}{2} = 1 \)). The magnitude of the fraction \( \frac{2}{4} \) is the same as the magnitude of the fraction \( \frac{1}{2} \). The fraction number \( \frac{2}{4} \) is just another name for the fraction number \( \frac{1}{2} \).

A more commonly used skill that involves renaming fractions is **simplifying fractions**. Here, students find an equivalent fraction in a specific format; the process is usually expressed as finding the equivalent fraction number in simplest form. This process can be thought of as the reverse of renaming.

#### Example

To express the fraction number \( \frac{3}{6} \) in simplest form, write out factors to find a common number in the numerator and denominator: \( \frac{3}{6} = \frac{1 \times 3}{2 \times 3} = \frac{1}{2} \). The number common to both numerator and denominator is 3.

The fraction \( \frac{7}{21} \) can be expressed as \( \frac{1 \times 7}{3 \times 7} \), which leads to the simplified form \( \frac{1}{3} \). The number common to both numerator and denominator is 7.

An efficient way to simplify fraction numbers is to see whether the prime numbers are common factors in the numerator and denominator. Start with 2, then try 3, 5, 7, and so forth. You don't have to find the largest common factor when using the prime factorization method to express a fraction number in simplest form; you can repeat the process until you reach the simplest form.

#### Example

Express the fraction number \( \frac{18}{90} \) in simplest form.

\[
\frac{18}{90} = \frac{9 \times 2}{45 \times 2} = \frac{9}{45} \\
\frac{9}{45} = \frac{3 \times 3}{15 \times 3} = \frac{3}{15} \\
\frac{3}{15} = \frac{1 \times 3}{5 \times 3} = \frac{1}{5}
\]

Therefore, the simplest form of \( \frac{18}{90} \) is \( \frac{1}{5} \).

Learners may misunderstand the process of simplifying fraction numbers. In part, confusion can arise from using the word **reduce** rather than **simplify** to describe the process. The word **reduce**, with its meaning “to make smaller,” might lead
Developing-Level Fraction Numbers

Developing-Level Fraction Numbers

Learners to believe the resulting fraction number has a smaller magnitude than
the original fraction number. This is not correct. The original fraction number and
the simplest form of the fraction number have the same magnitude. They are
just expressed differently. When exploring the concept of equivalent fractions,
therefore, be sure to say simplifying fractions rather than reducing fractions to help
the learner understand the process.

Exploring equivalent fraction numbers gives learners an opportunity to
demonstrate confidence and maturity in their fraction number sense. When
learners can comfortably decide what fraction numbers are equivalent, they can
usually identify any fraction, no matter how large or small. Unfortunately, many
misunderstandings and errors are possible when exploring equivalent fraction
numbers. These misunderstandings are discussed later in this section.

As the learner becomes more comfortable with fraction numbers and their
equivalents, you can expand the group of fraction numbers being investigated.
Building on the understanding of $\frac{1}{4}$ s, $\frac{1}{3}$ s, and $\frac{1}{2}$ s, have learners explore
$\frac{1}{5}$ s, $\frac{1}{6}$ s, $\frac{1}{8}$ s, $\frac{1}{10}$ s, $\frac{1}{12}$ s, and $\frac{1}{16}$ s. Why these fraction numbers, in particular?

Some ($\frac{1}{8}$ and $\frac{1}{16}$) we commonly use in measuring length and capacity; others
are useful because they easily translate into decimal numbers and percents. A
complete understanding of this expanded set of fraction numbers prepares the
learner for study of more advanced fraction numbers, as well as of decimal
numbers and percents. A learner does not need to know every possible
fraction combination (which would be impossible, anyway) but rather to grasp
the properties of fraction numbers, which can be applied to any fraction value.
By studying in depth a small group of fractions and their properties, therefore, a
learner should be able to apply these properties to any fraction number.

A number line is a good way to illustrate many properties of fractions, and the
classroom teachers in our focus groups confirmed that this simple instructional
aid is an effective teaching strategy. Using a number line also follows the
general strategy for concept development: start with the concrete mode, move
to the pictorial mode, and end with the abstract mode. And that correspondence
leads to the third principle for teaching developing-level fraction numbers.

Developing-Level Principle 3

Use a number line to represent fraction and decimal numbers; this technique
builds appropriate understanding.

The number line is a great tool for illustrating equivalent fraction numbers.
Example

On the number line show the fraction number $\frac{6}{8}$.
On another number line show the fraction number $\frac{3}{4}$.

![Number line with fractions marked: $\frac{6}{8}$ and $\frac{3}{4}$]

The locations of $\frac{6}{8}$ and $\frac{3}{4}$ are the same. So, $\frac{6}{8}$ and $\frac{3}{4}$ are equivalent fraction numbers.

You can also use the number line to order fraction numbers from largest to smallest, or smallest to largest. Finally, the number line can help learners locate fractions relative to a benchmark, such as the fraction $\frac{1}{2}$.

Examples

Order the following fraction numbers from smallest to largest.

$\frac{7}{8}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, $\frac{3}{8}$

Use a number line to order these fraction numbers.

![Number line with fractions marked: $\frac{1}{6}$, $\frac{3}{10}$, $\frac{3}{8}$, $\frac{5}{6}$, $\frac{7}{8}$]

Locate the fraction number $\frac{13}{22}$ using the benchmark $\frac{1}{2}$.

![Number line with fractions marked: $\frac{13}{22}$ and $\frac{1}{2}$]

A number line is simple to use, easy to construct, and always available to the learner for verification and clarification.

When investigating fraction values on the number line, it’s natural to explore the fraction numbers we often use in measurement, particularly in length and capacity.
Examples

• Cut a piece of string $2 \frac{1}{8}$ inches long.
• Measure $1 \frac{1}{2}$ cups of flour for a bread recipe.
• Fill a container $\frac{1}{2}$ full of water.
• Measure $\frac{1}{4}$ teaspoon of salt for making oatmeal.
• Double the recipe: $1 \frac{1}{3}$ cups of rice.

Each of these examples requires a solid fraction number sense, including an understanding of equivalency.

Activities with the number line and the customary measurement system naturally lead to investigating decimal numbers and their relationship to fraction numbers. These two mathematical entities should be explored concurrently, or at least consecutively, with fraction numbers studied first and closely followed by the study of decimal numbers.

Decimal Number Strategies

When learners become confident working with fraction numbers, it’s time to introduce decimal numbers. Most teaching of decimal number sense occurs in the intermediate years, in fourth, fifth, and especially sixth grade. However, it is also appropriate to emphasize decimal numbers in the seventh and eighth grade mathematics curriculum. Since decimal numbers look like whole numbers, many teachers falsely assume that they are easy to understand. In fact, many learners have a hard time with the concept of decimal numbers.

A guiding strategy for teaching decimal numbers focuses on these four questions (NCTM, 2000):

• What are decimal numbers?
• How are decimal numbers represented?
• How are decimal numbers related to whole numbers?
• How are decimal numbers related to fraction numbers?

These four questions lead to a conceptual understanding of decimal numbers. A learner must have whole number sense, place value confidence, and insight to understand decimal numbers as fraction numbers whose denominators are powers of 10 (10, 100, 1000, etc.). A special case occurs when the denominator is 100, which yields a direct correspondence of fraction numbers and decimal numbers to percents.

As usual, follow the general teaching sequence of concrete mode followed by pictorial mode followed by abstract mode when introducing decimal numbers. It is, admittedly, difficult to use concrete or hands-on materials to explore decimal numbers. The pictorial mode, on the other hand, is easy to incorporate.
The number line offers a great visual for studying decimal expressions. For instance, number lines can be constructed variously to illustrate the density property of decimal numbers: between any two decimal numbers there is another decimal number.

Example
Show the approximate location of 0.78 on the number line.

```
0 1
0.78
```

Show the approximate location of 0.07 on the number line.

```
0 0.1
0.07
```

Show the approximate location of 0.004 on the number line.

```
0 0.01
0.004
```

The number line also helps learners compare and order decimal numbers. The process is similar to comparing two whole numbers.

Example
Which value is greater, 1.62 or 1.48?
Show the approximate location of the decimal numbers on the number line.

```
0 1 2
1.48 1.62
```

There are two general approaches to developing learners’ decimal number sense. The first builds on learners’ knowledge of fraction numbers, the second on their knowledge of place value. Classroom teachers have indicated that using fraction knowledge seems to be more effective in helping the learner grasp the concept of decimal numbers.

**Fraction Approach**
An excellent visual approach that is a natural link to students’ fraction knowledge uses grid paper, usually a 10 x 10 square grid, to show decimal expressions.
Example

Show the decimal number 0.36 on the grid paper.

What portion of the graph is shaded? \( \frac{36}{100} = \text{thirty-six hundredths} = 0.36 \)

You can also show the link between fraction numbers and decimal numbers using more abstract teaching modes. Point out to students that decimal numbers are fraction numbers written in a different symbolic notation.

Example

The decimal number 0.68 should be read as "sixty-eight hundredths" (not "point six eight"), which clearly indicates its relation to fractional notation \( \frac{68}{100} \).

The decimal number 2.7 should be read as "two and seven-tenths" (not "two point seven"), just as you would read \( 2 \frac{7}{10} \).

Any fraction number can be expressed as a decimal number by applying the operation of division. In some cases, learners can complete this division operation mentally. For instance, given the fraction \( \frac{1}{2} \), a learner should recognize 0.5 as the corresponding decimal number (1.0 ÷ 2). Learners should be able to mentally calculate the corresponding decimal numbers for the following fraction numbers: \( \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \). They should also be able to express fraction numbers with denominators of 10, 100, or 1000 as decimal numbers. Thus, learners should recognize 0.3 as the corresponding decimal number of the fraction \( \frac{3}{10} \), and 0.40 as the corresponding decimal number of the fraction \( \frac{40}{100} \). With more complex fraction numbers, such as \( \frac{18}{37} \), learners can use a calculator to express the corresponding decimal number. Some complex fraction numbers, when expressed as decimal numbers, require a determination of significant digits. These more complex fraction numbers and corresponding decimal numbers will be discussed in the Advanced-Level section.
Place Value Approach

You can also use the place value approach to introduce decimal numbers to your students. The sequence of instruction begins with the whole number model, using it to explain the decimal number.

Example

For the whole number 3 4 5 6

6 is in the ones place
5 is 10 times the ones place, or in the tens place
4 is 100 times the ones place, so it is in the hundreds place
3 is 1000 times the ones place, so it is in the thousands place

Now continue to use the same general format to explain the decimal number using the place value approach.

Example

0 is in the ones place.
4 is \( \frac{1}{10} \) the ones place, so it is in the tenths place
5 is \( \frac{1}{100} \) of the ones place, so it is in the hundredths place
6 is \( \frac{1}{1000} \) of the ones place, so it is in the thousandths place

\[ 0.456 \]

Or,

\[ \frac{4}{10} + \frac{5}{100} + \frac{6}{1000} = 0.456 \]

To incorporate the pictorial teaching mode, you can show a place-value chart that includes both whole and decimal numbers. With the chart, learners can see at a glance the similarities and differences between whole number expressions and decimal expressions.

Example

The value 6783.129 is shown on the place value chart like this:

<table>
<thead>
<tr>
<th>Thousands</th>
<th>Hundreds</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>9</td>
</tr>
</tbody>
</table>

Note the place value chart is symmetric about the ones place.

The correct reading of the whole numbers is the sequence from left to right using the appropriate values, that is, thousands, hundreds, tens, and ones. So, we read 6,783 as “six thousand seven hundred eighty-three.” The correct reading of the decimal numbers also moves from left to right using the appropriate values of tenths, hundredths, and thousandths: “and one hundred
Developing-Level Fraction Numbers

Twenty-nine thousandths.” That last sentence is critical: it is imperative that learners practice reading decimal numbers by stating the quantitative value of the digits rather than using the word point and then reading the numerals.

Examples

0.6 is read six-tenths (not “point six”)
0.78 is seventy-eight hundredths (not “point seven eight”)
1.45 is one and forty-five hundredths (not “one point four five”)

Learners need to practice, practice, and practice some more reading decimal numbers to build their confidence. Include decimal numbers both larger than 1 (e.g., 3.45) and smaller than 1 (e.g., 0.56) in learners’ practice. Remind learners that the decimal symbol is read “and.” The number 3.45 is read “three and forty-five hundredths.”

The place value approach to studying decimal numbers also allows you to discuss the additive quality of both whole numbers and decimal numbers.

Examples

The whole number 45 can be expressed as 40 + 5 or 4 tens and 5 ones.
The decimal number 0.52 can be expressed as 0.5 + 0.02 or 5 tenths and 2 hundredths or 0.50 + 0.02 or 50 hundredths and 2 hundredths.

Learners can determine the relative magnitude of two decimal numbers by comparing each place value, beginning on the left and moving to the right. This is similar to comparing two whole numbers.

Examples

• Which value is greater? 245 or 685
  2 hundreds are less than 6 hundreds; so 685 is greater. No need to check the other digits.
• Which value is greater? 1.62 or 1.48
  The ones are the same value; 6 tenths are more than 4 tenths—so, 1.62 is greater. No need to check the other digits.

When learners are asked to compare more than two decimal numbers by ordering them from least to greatest or greatest to least, an efficient strategy is to take them all to the same place value.
Example
Order from least to greatest
1.07 0.08 0.38 0.0057 0.0406

Write all decimal numbers as ten thousandths, to match the smallest fraction of the group.
1.0700 0.0800 0.3800 0.0057 0.0406

Ordered from least to greatest
0.0057 0.0406 0.0800 0.3800 1.0700

Show the approximate location of each of the decimal numbers on a number line.

Learners can use some of the same strategies for comparing and ordering decimal numbers that they used to compare and order fraction numbers.

Understanding decimal numbers is critical to effectively grasping metric measurements, since the system is based on the powers of 10. Most scientific disciplines measure length in metric centimeters, meters, and kilometers, which are expressed as decimal numbers. We also use metric measurements, and decimal values, for mass (weight) and capacity. Students need to have a solid understanding of decimal numbers to convert from one unit to another in the metric system.

Example
A table measures 136 centimeters long. How many meters is this?

1 meter is 100 centimeters, or 100 centimeters = 1 meter
136 centimeters = unknown meters
\[
\frac{136}{100} = \text{unknown meters (divide by 100)}
\]
So, 136 centimeters is 1.36 meters

Exploring values using metric measures gives learners the chance to express decimal numbers in many different formats while reinforcing the fact that they are all equivalent (the object is the same length no matter how you change the decimal number). Practicing with decimal numbers helps learners understand the metric system, and vice versa.
Percent Strategies

Just as you can develop learners’ decimal number sense from their fraction number sense, you can start with fraction numbers to develop learners’ understanding of percents. It is a good idea to use learners’ knowledge of both fraction numbers and decimal numbers to introduce percents, however. Percents are tricky, so be sure to establish clear instructional goals: learners should be able to use percents appropriately and in the correct situations, to reason mathematically and explain why the answer is reasonable, and to recognize percents from contexts outside of the school setting. Before launching into more complex percent situations, review the meaning of the word percent: per means “by” and cent means “hundred”—“by the hundred.” Therefore, a percent represents a part-to-whole situation, where the whole is always 100. The 10 x 10 square grid, with its 100 squares, is thus a natural bridge into studying percents.

Example

Seven parts per hundred is $\frac{7}{100}$ or 7% or 7 percent.

The percent symbol % has two “zeros” and the slash mark, so it looks a bit like the number 100.

Using the 10 x 10 grid, the learner can explore percents, with each square representing 1%. By shading in the value of the percent on these 100 squares, the learner gets a visual reinforcement that percents are fraction numbers with denominators of 100.

Examples

- Show 37% on the 10 x 10 grid.
- Show 63% on the 10 x 10 grid.

37% and 63% complete the entire 10 x 10 grid—adding them together equals 100%.

The 10 x 10 grid allows the learner to practice identifying the relative magnitude of different percent values. The percents for this practice can be both less than and greater than 100%.
Example

On three different 10 x 10 grids, shade in the value of 46%, 21%, and 91%.
Use the grids to determine the order of these percents, from smallest to largest.

21% 46% 91%

Now use new grids and shade in the value of 158%.

A percent number line, similar to the number lines used in the study of fraction and decimal numbers, is a good tool for exploring percents. It’s also easy to make or draw. In certain situations, you may want to use multiple number lines to help students compare the magnitude of a group of percents. Also, students can learn a common set of benchmarks for estimating where percents appear on the number line. The benchmarks most appropriate for learners at the developing stage of percent knowledge are 10%, 50%, and 100%. Later you can expand these benchmarks to include 1% and 25%.
Examples

Locate these values on the percent number line:
37%  82%  8%  62%

Order these percent values from least to greatest on a number line:
63%  12%  56%  41%  89%

As for applying percents mathematically, students should learn how to find the percent of a number. This is quite enough at the developing stage of understanding percents. Determining percent is a common experience for many learners since discounts, sales taxes, and tips involve percents.

Example

Find 7% of $100.
$100 is called the base.
7% is called the rate and is expressed as a decimal.
Formula: base \times rate = percentage, or 100 \times 0.07 = 7
The result, $7, is called the percentage.

Example

Find the value of a tip of 20% on a meal costing $60.
base: $60
rate: 20% or 0.20
The tip is calculated as $60 \times 0.20 = 12$, or $12.

Later, you can introduce learners to other situations in which they would need to find the percent one number is of another number.

In teaching percents, be sure to focus on the relationship among fraction numbers, decimal numbers, and percents, which are closely aligned. A logical instructional sequence would be to have learners express

- fraction numbers as percents,
- decimal numbers as percents,
- percents as fraction numbers, and
- percents as decimal numbers.
Examples

Fraction Numbers  →  Percents

How would you express $\frac{3}{5}$ as a percent?
First express $\frac{3}{5}$ as a fraction number with 100 as denominator: $\frac{3}{5} \times \frac{20}{20} = \frac{60}{100}$.
So $\frac{3}{5}$ is equal to 60%.

How would you express 0.13 as a percent?
The decimal expression can be written as an equivalent decimal in hundredths. 0.13 can be expressed as 13% because $0.13 \times 100 = 13$.

Learners can also express fraction numbers as percents by way of decimal numbers: The fraction number $\frac{3}{5}$ expressed as a decimal is 0.60. This leads to $0.60 \times 100 = 60$ or 60%.

Examples

Percents  →  Fraction Numbers and Decimal Numbers

How would you express 12% as a fraction?
Percents are fraction numbers with denominators of 100, so 12% is expressed as $\frac{12}{100}$, or in simplified form as $\frac{3}{25}$.

87.5% is expressed as $\frac{875}{1000}$, or in simplified form as $\frac{7}{8}$.

How would you express 7.5% as a decimal?
Since a percent is a fraction with a denominator of 100, divide 7.5 by 100 to get the decimal number: $7.5 \div 100 = 0.075$. So, 7.5% expressed as a decimal number is 0.075.

You cannot overemphasize the relationship of fraction numbers, decimal numbers, and percents during the developing stage. Learners will acquire a more solid foundational understanding of each of these mathematical ideas when the instruction consistently demonstrates their relationships.

Informal Diagnostic Strategies

Incorrect assumptions regarding fraction numbers, decimal numbers, and percents at these early stages of learning can hinder students’ successful application of these numbers later on. That is why using an informal paradigm—detect, diagnose, prescribe, assess, and evaluate—to diagnose and correct these misunderstandings is key.
This paradigm is a cycle, which starts when you detect an error pattern. The next step, diagnosis—discovering what specific conceptual error is at play—is probably the most important. After determining the error, you can prescribe specific intervention activities. Finally, be sure to assess the results and evaluate whether the misunderstanding has been corrected. If the misunderstanding still exists, start the cycle again at diagnosis: perhaps the learner is having trouble with a different concept, or if not, perhaps he or she simply needs more practice.

Common Errors

Here are some of the common errors you may diagnose among your students. Often misconceptions about fraction numbers relate to the distinction between whole and parts in a fraction. Other errors arise from incorrect reasoning about equivalent fraction numbers. Errors in working with decimal numbers and percents include applying the procedures incorrectly or in the wrong sequence.

Example

Two pizzas are cut into six slices.

You ate one piece of each pizza. How much pizza did you eat?

Incorrect response: \( \frac{1}{12} \) of a pizza or \( \frac{2}{6} \) of two pizzas or \( \frac{2}{12} \) of one pizza

Incorrect reasoning: counting the number of pieces eaten without declaring the correct whole or unit

Correct response: \( \frac{2}{12} \) of two pizzas, where the two pizzas are considered the whole or unit
Example

What part of the cookie is shaded?
What part of the pizza is shaded?

Which shaded part gives you more food? Why?

Incorrect response: \(\frac{1}{2} = \frac{1}{2}\) or \(\frac{1}{2}\) cookie is the same as \(\frac{1}{2}\) pizza.
Incorrect reasoning: one half of anything is \(\frac{1}{2}\); disregards the size of the unit or whole.
One-half of a cookie is not usually equal in size to one-half of a pizza. The whole or unit is not understood in this context.

Correct response: \(\frac{1}{2}\) of the cookie is shaded, and \(\frac{1}{2}\) of the pizza is shaded, but \(\frac{1}{2}\) of the pizza gives you more food because the pizza is larger to begin with.

Example

Simplify
\[
\frac{4}{9}\quad \frac{3}{8}\quad \frac{3}{10}
\]

Incorrect response: \(\frac{2}{3}\quad \frac{4}{5}\quad \frac{1}{5}\)
Incorrect reasoning: All fraction numbers can be simplified using either 2 or 3 as the simplification factor.

Correct response: These fractions cannot be simplified further.

Example

Which decimal number is greater?
0.56 or 0.4321

Incorrect response: 0.4321 is greater.
Incorrect reasoning: The longer decimal (with more digits) is always greater.

Correct response: 0.56 is greater because 5 tenths are greater than 4 tenths.
Example
Which decimal number is greater?
0.2 or 0.31

Incorrect response: 0.2
Incorrect reasoning: Tenths are always greater than hundredths. The decimal number 0.2 (\( \frac{2}{10} \)) is greater than 0.31 (\( \frac{31}{100} \)).

Correct response: 0.31 is greater because 3 tenths are greater than 2 tenths, so there is no need to check the other digit.

Example
What number is 60% of 90?

Incorrect response: \( \frac{2}{3} \) or 0.67
Incorrect reasoning: \( \frac{60}{90} = \frac{20}{30} = \frac{2}{3} = 0.67 \)

Correct response: base x rate = percentage, so 90 x 60% (or \( \frac{60}{100} \) or 0.60) = 54

These are some of the common errors you’ll encounter among your students, but you’ll see many other errors as well. That is why we recommend applying the informal diagnostic cycle, so your students will encounter success as they work with fraction numbers, decimal numbers, and percents.

Tips for Success
These instructional tips will help your students develop number sense in fractions, decimal numbers, and percents. Use them as starting points and expand upon them as appropriate to your learning environment.

• Consistently remind the learner that fractions are numbers that have magnitude.
• Use geometric models to demonstrate fraction numbers, decimal numbers, and percents. One of these models is the 10 x 10 grid.
• Construct place-value charts to represent both whole numbers and decimal numbers.
• Use a number line to represent fraction numbers and decimal numbers to build appropriate understanding.
• Create equal sharing activities for learners to explore fraction equivalents.
• Remove the word reduce from the mathematical vocabulary. Replace it with the word simplify.
• Have learners practice reading decimal numbers aloud without using the word point to express the decimal symbol.
• The concept of equivalence of fraction numbers develops over time. Give your students plenty of time to learn.
• Don’t introduce high-value fraction numbers, decimal numbers, and percents at this stage: use common and low-magnitude examples to build a firm conceptual understanding of the complex relationship among these three terms.

• Use this sequence of instructional modes: concrete to pictorial to abstract, or touch it, see it, think it.

Protocols

A successful program involving fraction numbers, decimal numbers, and percents requires teachers to monitor learners’ progress during practice and to intervene between the practices as needed. Use the following protocols as intervention activities. Each protocol identifies a specific topic and presents pre-assessment information, the appropriate development level, a sequence of actions, and an assessment strategy for evaluation. You’ll also find suggestions for adapting the protocols for special needs, gifted, and English language learners.

The protocols are designed to take 20 to 30 minutes. The Launch section, which describes any preparation needed and instructions for teachers, may take 5 minutes; the Explore section, which describes what the students will do, 15 minutes; the Closure section 2 minutes and the Assessment and Evaluation sections together another 5 minutes. Of course, the protocols are quite flexible; you may modify them as needed for the most effective use in your particular learning environment. The time frame of each protocol can easily be adapted to any learning environment as well.

The protocols in this Developing-Level section focus on fraction numbers, decimal numbers, and percents. Some of the protocols are appropriate for small groups of two to four learners, and some involve the entire class. Several can be completed by an individual learner. In the case of the small groups, depending on the protocol, you may be able to present the protocol as either a learning activity or as a competitive game. Sometimes learners are stimulated by the gaming aspect of learning; other times, the competitive nature of winning or losing can detract from the learning environment. All the protocols require minimal materials. You may need to create some of the materials, but they are simple to construct: reproducible forms of specific instructional aids are provided in the appendix.
To find interventions suitable for your learners, refer to the matrix below.

## Developing-Level Protocols

<table>
<thead>
<tr>
<th>Title</th>
<th>Topic</th>
<th>Instructional Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marker Cover Up</td>
<td>greater than, less than, and equal to (fractions)</td>
<td>small group</td>
</tr>
<tr>
<td>Equivalent Match</td>
<td>equivalent fractions</td>
<td>small group</td>
</tr>
<tr>
<td>Make Equivalent Fractions</td>
<td>equivalent fractions</td>
<td>individual, small group</td>
</tr>
<tr>
<td>Numerator and Denominator</td>
<td>forming fractions based on numerator and denominator</td>
<td>small group</td>
</tr>
<tr>
<td>Decimal Number Cover</td>
<td>greater than, less than, and equal to (decimals)</td>
<td>small group</td>
</tr>
<tr>
<td>Checking 0.50</td>
<td>sorting decimals by magnitude</td>
<td>individual, small group</td>
</tr>
<tr>
<td>Decimal Number Comparison</td>
<td>decimal magnitude</td>
<td>small group</td>
</tr>
<tr>
<td>Decimal Names</td>
<td>decimal magnitude and decimal names</td>
<td>small group</td>
</tr>
<tr>
<td>Money Bags</td>
<td>relationship of fractions and decimals, leading to percents</td>
<td>whole group, small group</td>
</tr>
<tr>
<td>Percent Number Line</td>
<td>locating magnitude of percent on a number line</td>
<td>whole group, individual</td>
</tr>
<tr>
<td>Percents with Interlocking Circles</td>
<td>estimating percents on Interlocking Circles</td>
<td>whole group, small group</td>
</tr>
<tr>
<td>Grid Percent</td>
<td>different ways to represent percents</td>
<td>whole group, individual</td>
</tr>
</tbody>
</table>
### Marker Cover Up

**Topic:** Greater than, less than, and equal to  
**Level:** Developing  
**Instructional Mode:** Small group (two learners)  
**Time:** 20–30 minutes

#### Pre-assessment
- Check for understanding of fraction numbers using multiple models for the whole.
- Check for understanding of greater than, less than, and equal to for fraction numbers.

#### Materials
- a Marker Cover Up game board for every two learners  
  *(A reproducible form is provided in the appendix.)*
- one symbol cube showing the symbols <, >, and =; each symbol is represented on two faces
- one fraction cube showing these fractions on its faces: \(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\)
- two sets of 10 tokens to be used as markers. Each learner gets a set. The sets should be different colors so they can identify the player. The tokens might be circular counters, colored paper, or play money.

#### Launch
- Explain that this is a game or activity for two players.
- Each player rolls both the fraction cube and the symbol cube.
- The object of the activity or game is to find a fraction number on the Marker Cover Up game board that satisfies both the fraction cube and the symbol cube. This fraction number is covered by one of the player's tokens.

#### Explore
- Begin by rolling the two cubes and stating the conditions shown on the faces of the cubes.
- For example, if the symbol cube showed > and the fraction cube showed \(\frac{1}{3}\), what are some fraction numbers on the Marker Cover Up game board that would satisfy the condition “greater than one-third”? One possible answer is \(\frac{5}{6}\).
- Now determine who starts the activity by rolling the fraction cube. The player with the larger fraction goes first.
- Take turns rolling both the symbol cube and the fraction cube.
- The player covers the fraction number that satisfies the condition of the two cubes.
• If there is not a fraction number available that will satisfy the condition, the player must take a pass and the other player rolls the two cubes.

Closure
• The first player to complete a row, column, or diagonal line with five tokens can be considered the winner.
• An option is to continue playing until all the fraction squares on the game board are filled, without declaring a winner.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have the learner roll both cubes.</td>
<td>Success will be shown when the learner can quickly and accurately state at least two fractions that would satisfy the conditions on the cubes.</td>
</tr>
<tr>
<td>Ask the learner to state at least two fractions that would satisfy the conditions on the cubes.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Review the meaning of greater than, less than, and equal to.
• Explain what is meant by covering a row, column, or diagonal.

**Special Needs**
• Reduce the number that is needed to declare a winner from five markers to three.
• Allow two different fractions to be covered for every roll of the cubes. This would increase the pace of the activity.

**Gifted**
• Have learners create their own Marker Cover Up game boards.
• Have learners create their own fraction cubes. (Check that the conditions for the game are met—the game board shows equivalent fractions to the fractions on the fraction cube—with the learners’ fraction cube and Marker Cover Up game board.)
• Now have learners play the game.
Equivalent Match

**Topic:** Equivalent fractions

**Instructional Mode:** Small group (two learners)

**Level:** Developing

**Time:** 20–30 minutes

**Pre-assessment**
- Check that students understand when two numbers have the same magnitude.

**Materials (per group)**
- Fraction Number Cards
  
  *(A reproducible form is provided in the appendix.)*
- Remove the following 16 cards: \(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}, \frac{9}{12}, \frac{11}{12}\).

The deck now includes only 26 cards, which result in 13 equivalent matches.

**Launch**
- Explain the meaning of equivalence using two of the Fraction Number Cards, for example \(\frac{3}{12}\) and \(\frac{1}{4}\).
- Explain that the object of the activity is to find equivalent matches. You may present this as an activity or as a competitive game.
- Demonstrate the procedures:
  - Lay out the cards face down in rows.
  - Turn over two Fraction Number Cards and decide if they are equivalent fractions.
  - If they are equivalent, the player keeps the two Fraction Number Cards.
  - If they are not, the player turns them face down again in their original positions.

**Explore**
- Learners decide who starts by having each player turn over a Fraction Number Card. The one with the larger fraction begins the activity.
- Learners take turns picking two Fraction Number Cards, trying to find equivalent matches.
- Play continues until all the Fraction Number Cards have been matched.

**Closure**
- If presented as a game, the winner is the player with the most equivalent fraction matches.
- If used as an activity, no winner is declared. The activity could be completed more than once within the timeframe of the protocol.
Assessment

Have a learner turn over two Fraction Number Cards.

Ask if these two Fraction Number Cards are equivalents. If they aren’t, ask the learner to state some equivalent fractions to the two Fraction Number Cards drawn.

<table>
<thead>
<tr>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success is shown when learners recognize equivalent matches with increased accuracy and automaticity.</td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

**ELL**

- Explain the meaning of the words *equivalent*, *equivalence*, and *match*.
- Demonstrate the sequence of steps in playing the game.

**Special Needs**

- Reduce the deck to 20 or 16 Fraction Number Cards, with only 10 or 8 equivalent matches.
- As learners develop confidence, add some of the removed sets of equivalents back into the deck.

**Gifted**

- Put the Fraction Number Cards $\frac{3}{12}$, $\frac{4}{12}$, $\frac{8}{12}$, $\frac{9}{12}$ back into the deck.
- Learners should discover that some Fraction Number Cards do not have equivalent matches.
- Ask learners to name some equivalent matches for these remaining Fraction Number Cards.
Make Equivalent Fractions

**Topic:** Equivalent fractions

**Instructional Mode:** Small group, individual

**Level:** Developing

**Time:** 20–30 minutes

### Pre-assessment

- Learners should be able to make the $\frac{1}{2}$ shape with other fraction pieces from the Fraction Circles, for example, two $\frac{1}{4}$ Fraction Circles, or $\frac{2}{4}$, is equivalent to $\frac{1}{2}$.

### Materials

- Fraction Circles
  
  *(Reproducible circles are provided in the appendix.)*

- Choose the appropriate pieces to match the learners’ level of development.

### Launch

- Use Fraction Circles to demonstrate fractions that name the same number, and state that these are called equivalent fractions.

- For example, show that $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$.

### Explore

- Ask learners to find fraction pieces of only one color that can cover the $\frac{1}{2}$ shape. There will be more than one answer. They will be finding fractions equivalent to $\frac{1}{2}$.

- Ask learners to find only one-color fraction pieces that can cover the $\frac{1}{4}$ shape. These are the fractions equivalent to $\frac{1}{4}$.

- Ask learners to find only one-color fraction pieces that can cover the $\frac{1}{3}$ shape. These are the fractions equivalent to $\frac{1}{3}$.

- Ask learners to find only one-color fraction pieces that can cover the $\frac{1}{5}$ shape. These are the fractions equivalent to $\frac{1}{5}$.

### Closure

- Learners write the results of their equivalent fractions.

- For example, $\frac{1}{3} = \frac{2}{6}$ or $\frac{1}{3} = \frac{4}{12}$.
**Assessment**

Ask if the learner can create a rule for checking if two fractions are equivalent.

**Evaluation**

Success will be shown when learners can recognize that two fractions represent the same number, that is, the two fraction numbers are equivalent.

Success will be shown when the learner understands that a fraction is a number and that there are many fractions names for the same number.

---

**Adaptations for Various Learners**

**ELL**

- Explain the meaning of the words *equivalent* and *equivalence*.
- Demonstrate how to write equivalent fractions.

**Special Needs**

Have learners

- explore with the $\frac{1}{2}$ shape only until a high degree of confidence is shown.
- later explore with the $\frac{1}{3}$ shape until success is evident.

**Gifted**

Have learners

- choose another Fraction Circle piece to start.
- make a matrix showing all the fractions equivalent to this chosen Fraction Circle piece.
- continue to choose Fraction Circle pieces and add to their matrix.
- search the matrix for patterns or a general rule about equivalent fractions.
Numerator and Denominator

**Topic:** Forming fractions based on numerator and denominator

**Instructional Mode:** Small group (two, three, or four learners)

**Level:** Developing

**Time:** 20–30 minutes

**Pre-assessment**
- Check that learners can use a rectangular shape model to form a unit or whole.

**Materials**
- two number cubes with the following values on the faces: 1, 2, 3, 4, 6, 8
- a Rectangular Recording Sheet for each group *(A reproducible form is provided in the appendix.)*
- a different color marker or crayon for each learner in the group

**Launch**
Explain the procedures. You may present this as either an activity or a game.
- Each learner rolls the two number cubes, then forms a proper fraction using the numbers rolled (one number being the numerator and the other the denominator). For example, if a student rolls the values 3 and 8, the proper fraction would be \(\frac{3}{8}\).
- Next, the learners color (or cover) that many fractional parts on the Rectangular Recording Sheet. For \(\frac{3}{8}\), the learner could color three parts of the eight-part rectangle or one part of the four-part rectangle and one part of the eight-part rectangle.
- The object of the game or activity is to color or cover the maximum number of rectangular parts on the Rectangular Recording Sheet.

**Explore**
- Determine which learner starts by rolling one die. The learner who rolls the largest value begins the activity.
- The learner rolls both number cubes and decides how to color the proper fraction on the Rectangular Recording Sheet.
- Learners take turns rolling the number cubes and recording the proper fraction.
- Continue the activity until all the rectangles have been covered or colored.
Closure
• If this is an activity, the learners can count how many fractional parts of the rectangles they covered or colored.
• If this is a game, use the following scoring scheme:
  - Five points for each rectangle completely covered or colored
  - Two points for a rectangle that is more than $\frac{1}{2}$ colored
  - Zero points for all rectangles that are less than $\frac{1}{2}$ colored by the learner

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask the learner to roll both number cubes.</td>
<td>Learners have achieved success when they do this activity with confidence and accuracy, including creatively using the fractional parts to color the rectangles most efficiently and effectively.</td>
</tr>
<tr>
<td>Have the learner form the proper fraction.</td>
<td></td>
</tr>
<tr>
<td>Ask how the fractional parts could be used to color rectangles.</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
• Explain the meaning of the terms proper fraction, fractional parts, rectangle, and rectangular shape.
• Use only one rectangle on the Rectangular Recording Sheet for this activity: learners will need to color the proper fraction shown on one rectangle.

Special Needs
• Use only one rectangle on the Rectangular Recording Sheet.
• Present this as an activity, not a game: don’t use the scoring scheme, but rather have learners count how many fractional parts were colored.

Gifted
• Encourage more creative coloring of the rectangles. For example, if the proper fraction is $\frac{2}{3}$, the learner could color $\frac{1}{3}$ of one rectangle and $\frac{2}{6}$ of another.
Decimal Number Cover

Topic: Greater than, less than, and equal to
Level: Developing
Instructional Mode: Small group (two learners)
Time: 20–30 minutes

Pre-assessment
• Check for general understanding of the meaning of greater than, less than, and equal to.

Materials
• a Decimal Number Cover game board for every two learners
(A reproducible form is provided in the appendix.)
• one symbol cube with the symbols <, >, and = each on two faces
• one decimal cube with these decimal numbers on the faces: 0.25, 0.33, 0.50, 0.67, 0.75, 1.00
• two different-colored sets of tokens, approximately 10 tokens for each learner. These tokens can be circular counters, colored paper, or play money.

Launch
• Explain that this is a game or activity for two players.
• Each player rolls both the decimal and the symbol cubes.
• The object is to find a decimal number on the game board that satisfies both cubes. The player covers this number with his or her token.

Explore
• Learners determine who goes first by rolling the decimal cube. The player with the larger decimal number starts the activity.
• The player starts by rolling the two cubes and stating aloud the conditions shown. For example, if the symbol cube shows > and the decimal cube shows 0.75, the learner would state, “greater than seventy-five hundredths.”
• The player covers the decimal number that satisfies the condition of the two cubes. In the example above, one correct answer would be 0.88. The learner would cover 0.88 with her token.
• If no decimal number satisfies the condition of the two cubes, the player must take a pass and the other player rolls the cubes.
• Learners take turns rolling the cubes and covering decimal numbers.
**Closure**

- The first player to complete a row, column, or diagonal line with five tokens can be considered the winner.
- If playing this as an activity, not a game, learners continue until all the decimal squares on the Decimal Number Cover board are filled.

<table>
<thead>
<tr>
<th><strong>Assessment</strong></th>
<th><strong>Evaluation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask the learner to roll both cubes.</td>
<td>Success will be shown when the learner can quickly and accurately state at least two decimal numbers that would satisfy the condition on the two cubes.</td>
</tr>
<tr>
<td>Ask the learner to say aloud at least two decimal numbers that would satisfy the condition set by the cubes.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**

- Review the meaning of greater than, less than, and equal to.
- Explain what covering a row, column, or diagonal line means.

**Special Needs**

- Reduce the number that is needed to declare a winner from five markers to three.
- Allow two different decimal numbers to be covered for every roll of the cubes to increase the pace of the activity.

**Gifted**

- Have learners create their own Decimal Number Cover game board.
- Have learners create their own decimal cube. (Check that the conditions for the game are met—the game board shows equivalent decimals to the decimals on the decimal cube—with the learners’ decimal cube and Decimal Number Cover game board.)
- Now have learners play their game.
Checking 0.50

Topic: Sorting decimals by magnitude
Instructional Mode: Individual, small group (four learners)
Level: Developing
Time: 20–30 minutes

Pre-assessment
- Check for understanding of the concept of one-half.
- Check for understanding that one-half is also 0.50.
- Check for general understanding of the meaning of greater than, less than, and equal to.

Materials
- a set of Decimal Number Cards for each individual or group of learners
- three Label Cards with the expressions < 0.50, = 0.50, > 0.50
(Reproducible forms for the Decimal Number and Label Cards are provided in the appendix.)

Launch
- Show the three Label Cards.
- Discuss what each represents.
- Choose a Decimal Number Card from the set. For example, choose 0.40. Ask: Is this decimal number less than 0.50, equal to 0.50, or greater than 0.50? (It is < 0.50.)
- If learners hesitate, review the fraction number related to 0.40: \(\frac{40}{100}\). Remind learners that 0.50 written as a fraction is \(\frac{50}{100}\). Ask: Which fraction number is smaller, \(\frac{40}{100}\) or \(\frac{50}{100}\)? (\(\frac{40}{100}\)). Therefore, 0.40 is less than 0.50.

Explore
- Lay out the three Label Cards horizontally in this order:
  < 0.50 = 0.50 > 0.50
- Place the Decimal Number Cards in a deck arrangement, face down.
- One learner selects the top card from the deck and places it under the correct Label Card.
- Learners take turns until all the Decimal Number Cards are arranged correctly.
- Repeat this activity for at least five cycles, using all the Decimal Number Cards each time.
• If a learner is unsure about where to place the Decimal Number Card, remind him or her that every decimal number can be expressed as a fraction number.

**Closure**
• Have all the learners work together to arrange the entire Decimal Number Card deck in order from smallest to largest.
• Discuss the magnitude of the decimal numbers in the Decimal Number Card set.

<table>
<thead>
<tr>
<th><strong>Assessment</strong></th>
<th><strong>Evaluation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Choose a Decimal Number Card.</td>
<td>Success will be shown when the learner can quickly determine if the decimal number is greater than, less than, or equal to 0.50.</td>
</tr>
<tr>
<td>Have the learner declare whether the decimal number is &lt; 0.50, = 0.50, or &gt; 0.50.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Review the meaning of greater than, less than, and equal to.

**Special Needs**
• Begin the activity with only seven Decimal Number Cards (the full set has 14).
• Use only the > 0.50 Label Card to practice.
• Later include the < 0.50 Label Card.
• As learner confidence is established, include all 14 Decimal Number Cards and all three Label Cards.

**Gifted**
• Have learners expand the Decimal Number Card deck to include expressions with three decimal places and complete the activity using this expanded deck.
• Combine the Decimal Number Cards and the Fraction Number Cards and complete the activity.
Decimal Number Comparison

**Topic:** Decimal magnitude

**Instructional Mode:** Small group (two learners)

**Level:** Developing

**Time:** 20–30 minutes

---

**Pre-assessment**
- Check for understanding of < 0.50, = 0.50, and > 0.50.
- Check for understanding of the concept of larger and smaller decimals.

**Materials**
- a set of Decimal Number Cards for each pair of learners
  
 >= Reproducible cards are provided in the appendix.

**Launch**
- Show a few decimal numbers from the Decimal Number Card deck.
- Select two Decimal Number Cards.
- Ask learners which decimal number is larger.
- Ask them to explain how they know which is larger.

**Explore**
- The Decimal Number Cards are shuffled and set as a deck face down between the two learners.
- Each learner picks a card from the deck.
- The learner with the larger decimal number takes both cards for that round.
- Repeat process until the entire set of Decimal Number Cards has been used.
- Play another game where the learner with the **smaller** decimal receives both cards for that round.
- If playing as a game, the learner with the most cards wins.
- If playing as an activity, no winner will be declared.

**Closure**
- Review what the largest decimal number in the deck of Decimal Number Cards is. (1.00)
- Review what the smallest decimal number in the deck is. (0.10)
- Explain some of the attributes of each of these decimal numbers.
**Assessment**
- Ask the learner to select two cards from the Decimal Number Card deck.
- Ask which decimal number is larger.
- Repeat with two other Decimal Number Cards, but this time ask which decimal is smaller.

**Evaluation**
- Success will be shown when the learner can correctly compare two decimal numbers and correctly declare which is larger (or smaller).

---

**Adaptations for Various Learners**

**ELL**
- Explain the meaning of larger and smaller decimals.

**Special Needs**
- Begin by using only 8 cards from the Decimal Number Card deck.
- Later include all 14 cards from the Decimal Number Cards.
- If learners are not sure which decimal number is larger, they can put those two cards on a “bone pile” for later discussion.

**Gifted**
- Have the learners expand the Decimal Number Cards to include decimal numbers with three places, or use the expanded Decimal Number Cards your students may have already constructed from the Checking 0.50 protocol.
- Learners play the game with the expanded Decimal Number Cards.
- Have learners arrange the expanded Decimal Number Card deck from smallest to largest in magnitude.
Decimal Names

**Topic:** Decimal magnitude and decimal names  
**Instructional Mode:** Small group (two learners)  
**Level:** Developing  
**Time:** 20–30 minutes

### Pre-assessment
- Check for understanding of the place values of tenths and hundredths (e.g., 0.3 and 0.31).

### Materials
- a set of Decimal Number Cards (14 cards)
- a set of Decimal Name Cards (14 cards), which correspond to the Decimal Number Cards  
  *(Reproducible sets of both are provided in the appendix.)*

### Launch
- Show a card from the Decimal Number Card set, for example, 0.30.
- Show a few cards from the Decimal Name Cards including the card “thirty-hundredths.”
- Find the corresponding Decimal Name Card for the selected Decimal Number Card.

### Explore
- Combine the Decimal Number Cards and the Decimal Name Cards into one deck.
- Lay the 28 cards into a 4 by 7 array, face down.
- Have one learner turn over two cards.
- If it is a Decimal Number Card that is matched with the corresponding Decimal Name Card, the learner keeps both cards. The learner must pronounce the Decimal Name Card. If successful, the same learner plays again. If not, the two cards are returned face down to their original position in the array.
- Learners continue taking turns looking for matches in a “Concentration”-type situation until the entire array is cleared.
- If playing as a game, the winner is the learner with the most matches.
- If playing as an activity, there is no winner.
- Repeat the game or activity by mixing the cards and resetting the array.
Closure

- Review how to pronounce the decimal names using one of the cards as an example.
- Have learners practice pronouncing decimal names using the Decimal Name Cards.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask the learner to select a Decimal Number Card from the set.</td>
<td></td>
</tr>
<tr>
<td>Have the learner look through the Decimal Name Cards to find the correct name.</td>
<td></td>
</tr>
<tr>
<td>Success will be shown when the learner can correctly match the Decimal Number Card with its corresponding Decimal Name Card.</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
- Explain the meaning of the words tenths and hundredths.
- Have the learner practice pronouncing a few of the Decimal Name Cards before beginning the game or activity.

Special Needs
- Begin the activity with only six Decimal Number Cards and the six corresponding Decimal Name cards.
- Have learners say the decimal name, if able, or just match the cards.
- Later expand the Decimal Number Cards and Decimal Name Cards to include the entire set.

Gifted
- Use the expanded Decimal Number Cards from Checking 0.50.
- Have learners make the corresponding Decimal Name Cards to match the expanded Decimal Number Cards.
- Learners play the game or activity with the expanded Decimal Number Cards and Decimal Name Cards.
Money Bags

**Topic:** Relationship of fraction and decimal numbers, leading to percents  
**Level:** Developing

**Instructional Mode:** Whole group, small group

**Time:** 20–30 minutes

---

### Pre-assessment
- Check for understanding of the relationship between fraction numbers and decimal numbers.
- Check for understanding of the basic coins in the money system.

### Materials
- a bag or container with 100 pennies

### Launch
- Lead a discussion about the relationship of a penny to a dollar (100 pennies), where the dollar is the unit or whole.
- Draw one penny from the bag.
- Remind students that one penny is \( \frac{1}{100} \) of the whole dollar. The corresponding decimal number is 0.01.
- Replace the penny and draw two pennies.
- How can this fraction number be expressed? \( \frac{2}{100} \)
- How can this decimal number be expressed? (0.02)

### Explore
- Have a learner remove some pennies from the bag.
- Have this learner count the number of pennies withdrawn.
- The other learners must write the fraction number and its corresponding decimal number.
  - For example, if 12 pennies were withdrawn, the fraction would be \( \frac{12}{100} \) and the corresponding decimal number would be 0.12.
- Continue replacing the pennies and having other learners draw pennies. The rest of the learners will write the fraction and corresponding decimal numbers. Sometimes ask a learner to state aloud the corresponding decimal name.
- Continue this activity until every learner has had an opportunity to withdraw and count a group of pennies.
**Closure**
- Lead a discussion about money, decimal numbers, and fractions.
- Keep the focus of the discussion on the relationship among fraction numbers, decimal numbers, and representations of money.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask: How many pennies are in a quarter? (25)</td>
<td>Success will be shown when learners can correctly represent the fraction number with a denominator of 100 and its corresponding two-place decimal number.</td>
</tr>
<tr>
<td>Ask the learners to represent a quarter as a fraction number (try to obtain the simplified fraction) and as a decimal number. ($\frac{1}{4}$) (0.25)</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
- Review the coins in the money system (penny, nickel, dime, quarter).

**Special Needs**
- When drawing pennies from the bag, limit the total drawn to no more than 20 pennies.
- Accept the fraction number given without simplification.
- Construct a two-place grid to facilitate writing the corresponding decimal number.

```
tenths  | hundredths  
0.      |             
```

**Gifted**
- Have learners draw more than 20 pennies.
- Have learners always simplify the fraction number.
- Place more than 100 pennies in the bag. Now ask learners to express the fraction number and corresponding decimal number.
Percent Number Line

Topic: Locating magnitude of percent on a number line
Instructional Mode: Whole group, individual
Level: Developing
Time: 20–30 minutes

Pre-assessment
• Check understanding of a basic number line and how to construct a number line.
• Check general understanding of benchmarks.

Materials
• percent number lines, constructed by the learners

Launch
Each learner will construct his or her own percent number lines:
• On an 8 1/2” x 11” sheet of paper, have the learners use their centimeter rulers to create six horizontal number lines.
• Each of these six number lines should be 20 centimeters. The number lines will just fit on an 8 1/2” x 11” sheet.
• Have the learners label each number line with 0%, 25% (5 centimeters), 50% (10 centimeters), 75% (15 centimeters), and 100% (20 centimeters). The labels should be written above the number line.

Explore
• On their percent number lines, have learners find the approximate location of each of the percents in the table below.
• The locations can be marked with a symbol on the number line and the percent value written below the symbol.

<table>
<thead>
<tr>
<th>Number Line</th>
<th>Percent Value</th>
<th>Percent Value</th>
<th>Percent Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>10%</td>
<td>90%</td>
<td>60%</td>
</tr>
<tr>
<td>#2</td>
<td>43%</td>
<td>67%</td>
<td>22%</td>
</tr>
<tr>
<td>#3</td>
<td>86%</td>
<td>2%</td>
<td>34%</td>
</tr>
<tr>
<td>#4</td>
<td>49%</td>
<td>98%</td>
<td>74%</td>
</tr>
<tr>
<td>#5</td>
<td>63%</td>
<td>77%</td>
<td>69%</td>
</tr>
<tr>
<td>#6</td>
<td>89%</td>
<td>51%</td>
<td>40%</td>
</tr>
</tbody>
</table>
Closure

• Lead a discussion about the value of the benchmarks (0%, 25%, 50%, 75%, and 100%).
• Keep the focus of the discussion on the benchmarks.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Give the learner the value 56%.</td>
<td>Success will be shown when the learner correctly locates the given percent value on a percent number line and can articulate percent values larger and smaller than the given value.</td>
</tr>
<tr>
<td>Have the learner locate this value on any of his or her percent number lines.</td>
<td></td>
</tr>
<tr>
<td>Have the learner say aloud some values larger than 56% and some values smaller than 56%.</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

**ELL**

• Explore the meaning of benchmarks by giving an example such as a traveling benchmark.
  Coming to school you pass a certain building; when you pass this building, you know it will be only a short while until you arrive at school. This building could be considered a benchmark on your journey to school.

**Special Needs**

• Give learners preconstructed percent number lines with the benchmarks labeled.
• Have learners locate only one of the three given percent values for each percent number line.
• Later, learners could locate two of the three given percent values for each percent number line.

**Gifted**

• Have learners construct a few percent number lines labeled to 200%.
• Now have learners locate percent values exceeding 100%. For example, locate 133% on the percent number line.
• Given the fraction $\frac{57}{100}$, ask learners to find the corresponding percent location on a percent number line.
Percents with Interlocking Circles

**Topic:** Estimating percents on Interlocking Circles  
**Level:** Developing  
**Instructional Mode:** Whole group, small group (two learners)  
**Time:** 20–30 minutes

**Pre-assessment**
- Check for understanding that the unit, or whole, is 100%, half is 50%, and one-fourth is 25%.

**Materials**
- Interlocking Circles of different colors  
  *(Reproducible circles are provided in the appendix.)*

**Launch**
- Show learners how to construct their own Interlocking Circles.  
- Practice rotating the Interlocking Circles.

**Explore**
Ask learners to show approximate percents on their Interlocking Circles. Use the values in the matrix below (the matrix, however, offers only a small sample of what can be shown using the Interlocking Circles).

| Show about 50% red.  
| Show about 30% blue.  
| Show about 27% red.  
| Show about 33% blue.  
| Show about 67% red.  
| Show about 10% red.  
| Show about 90% blue.  
| Show about 77% red.  
| Show about 21% red.  
| Show about 85% blue.  
| Show about 25% blue. When 25% is blue, about what percent is red?  
| Show about 60% red. When about 60% is red, about what percent is blue?  
| Show about 80% blue. About what percent is red?  
| Show about 50% red. About what percent is blue? |
• Have the learners form partners.
• One partner states the percent value and color that must be shown and the other partner demonstrates it with the Interlocking Circles.
• Learners take turns reading out and showing the percent values.

Closure
As a whole group,
• Ask learners to show 61% red on their Interlocking Circles.
• Ask learners to show 15% blue on their Interlocking Circles.
• Discuss how the Interlocking Circles show approximate percent values.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask learners: If about 30% red shows on the Interlocking Circles, approximately what percent is blue?</td>
<td>Success will be shown when the learner correctly estimates the complementary percent value (70%) on the Interlocking Circles.</td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Identify the red and blue colors on the Interlocking Circles.

**Special Needs**
• Use preconstructed Interlocking Circles.
• Make marks on the edges of the Interlocking Circles to assist learners in finding 25% and 50%.
• Ask learners to find at least four of the listed percent values on their own.
• When learners partner up, limit the percent values they must find.

**Gifted**
• Using more than one Interlocking Circles device, show percent values that exceed 100%.
• Explore the pattern of complementary percent values on the Interlocking Circles.
Grid Percent

**Topic:** Different ways to represent percents  
**Level:** Developing  
**Instructional Mode:** Whole group, individual  
**Time:** 20–30 minutes

**Pre-assessment**
- Check for basic understanding of fractions and decimals.

**Materials**
- Grid Percent Matrix Sheet  
(A reproducible form is provided in appendix.)  
- 10 x 10 Grid Sheet  
(A reproducible form is provided in appendix.)

**Launch**
- Discuss the meaning of the word **percent**. The word comes from the Latin *per centum*, “out of a hundred” or “parts out of a hundred.”
- Percents can be represented as a fraction with the denominator 100, as a decimal with two places (hundredths) and as a picture or diagram on a 10 x 10 grid.
- For example, \( \frac{30}{100} \) can be written as 0.30 or 30% or shown as 30 squares shaded on the 10 x 10 grid.

**Explore**
- Present learners with the Grid Percent Matrix Sheet.

### Grid Percent Matrix Sheet

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
<th>Picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{22}{100} )</td>
<td>0.22</td>
<td>22%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>( \frac{63}{100} )</td>
<td>0.63</td>
<td>63%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>( \frac{0.86}{100} )</td>
<td>0.86</td>
<td>86%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>( \frac{57}{100} )</td>
<td>0.57</td>
<td>57%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>( \frac{13}{100} )</td>
<td>0.13</td>
<td>13%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
</tbody>
</table>
• The matrix shows one way of four to name or show a value; the learners need to determine the other names for the value.
• Have learners complete the matrix and show each value on the grid sheet.
• If learners need more practice, expand the matrix by adding more values.

**Closure**
• Lead a discussion on how percents, decimal numbers, and fraction numbers are different ways to represent the same value.
• Keep the focus of the discussion on the different ways to represent the same value.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Give the learner the value (\frac{37}{100}). Ask what the corresponding decimal number is. (0.37) Ask what the corresponding percent is. (37%) Ask the learner to show a 10 x 10 grid with the value shaded in.</td>
<td>Success will be shown when the learner can quickly represent a fraction as a decimal number and also as a percent. In addition, the learner should accurately shade this value on a 10 x 10 grid.</td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Check for understanding of the terms fraction, decimal, percent, grid, shade, picture, and diagram.

**Special Needs**
• Have learners fill in only one other cell per row in the matrix.
• Later, have learners complete the other cells in the matrix.
• Assign the grid picture only after learners are confident in their ability to accurately shade the correct number of squares in the 10 x 10 grid.

**Gifted**
• Have learners create their own matrix of at least six values, with only one form shown per row.
• Have learners exchange their matrices with one another.
• Encourage learners to include challenging values in their matrices.
Advanced-Level Fraction Numbers

107  Fraction Number Strategies
112  Decimal Number Strategies
115  Percent Strategies
118  Informal Diagnostic Strategies
119  Common Errors
121  Tips for Success

122  Protocols
124  Fraction Number Comparison
126  Pick Your Fraction
128  Numerator and Denominator Challenge
130  Stack Them Up
132  Build a Biggie
134  Decimal Potpourri
136  Heads & Tails
138  Draw a Percent
Advanced-Level Fraction Numbers

The Basic-Level and Developing-Level sections of this guide contain strategies to give learners an enhanced foundation in fraction numbers. This section, focusing on advanced-level learners, builds upon that foundation to include addition and subtraction of fraction numbers, precision skills required for decimal numbers, and advanced situations involving percents. The Basic- and Developing-Level principles are also reinforced in this section.

### Build a Strong Foundation
If you are using this guide in an intervention setting, make sure your students have a solid understanding of the concepts in the basic and developing levels before working in the advanced level. If students struggle with the advanced level, it may be appropriate to begin at the basic or developing level.

#### Basic-Level Principle 1
Build upon knowledge students already possess: introduce fraction numbers by having students explore how to share a set of objects equally among a group.

#### Basic-Level Principle 2
Introduce division in the early grades: present sharing and the concept of division simultaneously.

#### Basic-Level Principle 3
Use equal sharing activities to introduce the concept of size in fraction numbers.
Developing-Level Principle 1
Emphasize that fractions are numbers, and they have magnitude.

Developing-Level Principle 2
Explore equal sharing activities to develop learners’ understanding of fractional equivalence.

Developing-Level Principle 3
Use a number line to represent fractions and decimal numbers; this builds appropriate understanding.

These Basic and Developing principles lead to the three principles underlying the Advanced-Level teaching strategies:

Advanced-Level Principle 1
Emphasize the concepts of significant digits and precision: learners must have an advanced number sense for a clear understanding of these complex topics.

Advanced-Level Principle 2
Teach estimating and benchmarking: they are powerful tools to help students understand fraction numbers, decimal numbers, and percents.

Advanced-Level Principle 3
Ensure a deep understanding of fraction numbers and their attributes before introducing addition and subtraction with fractions.

When teaching advanced-level fraction numbers, emphasize the operations of addition and subtraction. Also at this level, learners need a more complete understanding of significant digits and precision in relation to decimal numbers. To use a calculator, learners must be skilled at determining the number of significant digits in every numerical situation. In conjunction with the precision needed in certain situations, learners must be confident in matching decimal numbers to their corresponding fraction numbers. While a calculator can be a valuable tool in these situations, the learner must still determine the appropriate decimal number answer for the given context. Learners at this level must also understand the different ways a percent can be represented and calculated.
In this guide, adding and subtracting fractions is limited to those with like denominators. When fraction numbers with unlike denominators are involved, learners will use their estimating and benchmarking skills to arrive at an approximate answer. Learners will use a number line to assist in benchmarking in those situations. We also use the 10 x 10 grid of 100 squares to represent percents in this guide. A more advanced application of the 10 x 10 grid, presented in one of the protocols, links percents with an artistic diagram.

As with the Basic-Level and Developing-Level sections, this section is organized into six areas: fraction number strategies, decimal number strategies, percent strategies, informal diagnostic strategies, common errors, and tips for success. These strategies are not exhaustive; they are sample strategies that have proven successful for some advanced-level learners. The ten tips for success offer a quick reference of best practices for further developing students’ understanding of fraction numbers, decimal numbers, and percents.

Eight protocols appropriate for advanced-level study of fraction numbers, decimal numbers, and percents follow the discussion of general strategies; three focus on addition and subtraction, three on decimal numbers, and two on percents. The protocols are not complete lesson plans or curriculum units; instead, consider them templates with a specific focus that you can modify as needed to suit your day’s lesson plan. While designed primarily for small- and whole-group instruction, the protocols can be used for individual tutoring by an educational aide or parent.

**Fraction Number Strategies**

At the advanced level of instruction, teachers may wish to spend little time on the hands-on materials and diagrams in favor of the abstract mode. However, moving too quickly from the hands-on and diagram modes would not be appropriate. Learners need to experience the more complex fraction numbers following the same sequence of instruction used for simple fraction numbers. The Basic-Level and Developing-Level sections of this guide contain many instructional activities that incorporate the hands-on materials and diagrams that can be expanded to include more complex fraction numbers. While there will be fewer opportunities for using hands-on materials at this level, an ideal sequence of instruction for these fraction numbers would be at least a brief time with the hands-on materials followed by exploring simple diagrams, which would then lead to the abstract mode of expressing the fraction.

There will, however, be instances when the fraction expression does not lend itself to realistic hands-on activities or simple diagrams. In these cases a more formal definition of a fraction would be appropriate.
Fraction: A fraction is expressed as $\frac{a}{b}$, where $a, b$ are integers and $b \neq 0$.

Proper fraction: A proper fraction is every $\frac{a}{b}$ where $a < b$ (e.g., $\frac{1}{3}$, $\frac{2}{5}$, $\frac{5}{7}$).

Improper fraction: An improper fraction is every $\frac{a}{b}$ where $a > b$ or $a = b$ (e.g., $\frac{3}{2}$).

Mixed number: Any fraction greater than 1 can be written as a mixed number. A mixed number is a way of expressing a fraction number greater than 1 as a whole number and a fraction. For example, $\frac{6}{5}$ can be expressed as the mixed number $1 \frac{1}{5}$.

With these definitions advanced-level learners can explore and apply properties of fraction numbers to any fraction.

One practical application that illustrates the definition of a fraction is finding the slope of a line. The concept of slope can be used to represent fraction numbers as well as operations on fraction numbers.

**Slope of a Line**

A slope is related to a line by expressing the rise (vertical distance) and run (horizontal distance) as a fraction number. The rise will be the numerator of the fraction, and the run will be the denominator.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Understanding fraction numbers and their relationship to the slope of a line empowers learners studying algebra. Problems grasping fraction numbers and their operations will become a major obstacle to learners’ progress in most mathematics, but especially in algebra.

At the advanced level, learners move in a natural progression from studying single fraction numbers to applying the operations of addition and subtraction to them. Addition and subtraction are binary operations, which means they combine two entities into a single entity. This is not a minor intellectual jump: in fact, it is usually a major challenge for learners. In this guide we limit exploration to adding and subtracting fraction numbers with like denominators, and only benchmarking and estimating sums and differences of fraction numbers with unlike denominators (limiting the denominators to 5 and 25).

Adding or subtracting two fraction numbers with like denominators is similar to adding or subtracting two whole numbers. However, learners may stumble into misunderstandings if careful attention is not given to fraction properties. Therefore, before asking learners to add or subtract fraction numbers, be sure they are confident applying the fraction properties of equivalence, order, and magnitude. Beginning with the simple addition of two fraction numbers and
building to more complex expressions, the learner will gain confidence in adding and subtracting fraction numbers.

**Adding two fraction numbers with like denominators**

**First stage practice**

\[
\frac{1}{8} + \frac{3}{8} = ?
\]

Have the learner use hands-on materials or make a diagram of the situation.

\[
\frac{1}{8} + \frac{3}{8} = \frac{4}{8}
\]

Usually the final fraction expression is simplified.
What is the simplified expression?

\[
\frac{4}{8} = \frac{2 \times 2}{2 \times 4} = \frac{2}{4} = \frac{2 \times 1}{2 \times 2} = \frac{1}{2}
\]

Here we have shown the prime factorization method for simplifying fraction numbers.
Adding or subtracting fraction numbers with like denominators can result in fraction expressions that are greater than 1. The same sequence for adding or subtracting two fraction numbers is followed. The resulting fraction number can be expressed as an improper fraction or as a mixed number.

**Adding two fraction numbers with like denominators**

**Second stage practice**

\[
\frac{5}{9} + \frac{7}{9} = ?
\]

\[
\begin{array}{c}
\text{5 parts of 9} \\
\text{7 parts of 9} \\
\text{12 parts of 9}
\end{array}
\]

Note the unit is 9, not 18.

So, \( \frac{5}{9} + \frac{7}{9} = \frac{12}{9} \).

This fraction sum can be expressed in various ways. \( \frac{12}{9} \) can be simplified to \( 1 \frac{3}{9} \) or \( 1 \frac{1}{3} \) or \( 4 \frac{3}{3} \). Adding or subtracting fraction numbers is an ideal time to practice benchmarking to 1, \( \frac{1}{2} \), or 0.

**Example**

Is \( \frac{5}{9} + \frac{7}{9} \) greater than 1, less than 1, or equal to 1?

Both of these fraction numbers are greater than \( \frac{1}{2} \), so the sum of the two yields a fraction number greater than 1.

Learners at the advanced level must develop the key skill of using benchmarks. This skill will prove valuable in many different contexts, not only with fraction numbers but also later with decimal numbers.

**Subtracting two fraction numbers with like denominators**

\[
\frac{11}{15} - \frac{2}{15} = ?
\]

Benchmark each fraction to 1 or \( \frac{1}{2} \) or 0.

Draw a number line and locate these two fraction numbers.
Advanced-Level Fraction Numbers

Seeing the difference between the two fraction numbers (subtraction) on the number line, learners should be able to give an approximate resulting fraction number of \( \frac{1}{2} \).

Numerically, \( \frac{11}{15} - \frac{2}{15} = \frac{9}{15} \). Simplifying the result leads to \( \frac{3}{5} \), which is a little larger than \( \frac{1}{2} \), as estimated.

After sufficient practice with benchmarking and estimating as ways to add or subtract two fraction numbers with like denominators, learners should be ready to explore these operations with unlike denominators. As noted, in this guide we limit this practice to results that have denominators of 5 and 25.

**Estimating the result of adding two fraction numbers with unlike denominators**

\[
\frac{3}{5} + \frac{12}{25} = ?
\]

Locate each fraction on a number line.

The fraction \( \frac{3}{5} \) will be a little more than \( \frac{1}{2} \).

The fraction \( \frac{12}{25} \) will be a little less than \( \frac{1}{2} \).

Therefore, an estimate for the sum of these two fraction numbers is approximately 1.

Applying benchmark skills to many situations builds learners’ confidence before doing the routine procedure of the operations. Being able to estimate will also help learners check the actual fraction number sum or difference.

**Estimating the result of subtracting two fraction numbers with unlike denominators**

\[
\frac{17}{25} - \frac{2}{5} = ?
\]

Show the location of the two fraction numbers on a number line.

\( \frac{17}{25} \) is a little more than \( \frac{1}{2} \). \( \frac{2}{5} \) is close to \( \frac{1}{2} \) but less than \( \frac{1}{2} \).

The learner should determine that the result is less than \( \frac{1}{2} \), and maybe close to \( \frac{1}{4} \).
Adding and subtracting fraction numbers can be a challenge for many learners. If you approach these operations in stages, first working with like denominators and then using benchmarking, learners are more likely to succeed.

**Decimal Number Strategies**

The strategies for teaching decimal numbers at the advanced level reflect the common use of a calculator in math classes. Most calculators express values in a decimal format. With more complex decimal numbers, it is crucial for learners to have a clear understanding of the concepts of significant digits, rounding or approximate value, terminating, and repeating. Implicit in most of these concepts is the need to know just how precise the decimal number must be, based on the context. A few definitions will help learners determine the level of precision needed within a context.

**Terminating decimal number**

*When fully expressed, a terminating decimal number ends with only zeros following the last nonzero digit. Typically, these zeros are not shown.*

**Example**

- 0.25 (\(\frac{1}{4}\)) is a terminating decimal (zeros not shown)
- 0.5 (\(\frac{1}{2}\)) is a terminating decimal
- 0.375 (\(\frac{3}{8}\)) is a terminating decimal

**Repeating decimal number**

*When fully expressed, a repeating decimal number eventually repeats a group of digits.*

**Example**

- 0.3333... (\(\frac{1}{3}\)) is a repeating decimal
- 0.6666... (\(\frac{2}{3}\)) is a repeating decimal
- 0.416666... (\(\frac{5}{12}\)) is a repeating decimal
- 0.343434... (\(\frac{34}{99}\)) is a repeating decimal

The usual notation to show a repeating decimal number is a bar above the repeating numeral(s).

Therefore,

- 0.3333... is shown as \(0.\overline{3}\)
- 0.6666... is shown as \(0.\overline{6}\)
- 0.416666... is shown as \(0.4\overline{1}\)
- 0.343434... is shown as \(0.\overline{34}\)
Significant digits

The significant digits begin with the first nonzero digit and end with the last digit written.

Examples
321, 32.1, and 0.0321 all have three significant digits: 3, 2, and 1.
3210, 321.0, and 0.003210 all have four significant digits: 3, 2, 1, and 0.

Rounding off, or approximate values

A number is rounded off to a specific number of significant digits by simply dropping the digits after the target digit (rounding down, if the next digit is less than 5), or by raising the target digit by 1 (rounding up, if the next digit is 5 or greater) and dropping the remaining digits. To round 0.31627 to three significant digits, look at the fourth nonzero digit. Since it is less than 5, leave the third digit as is and drop the remaining digits: 0.316.

Examples
Round off to find the approximate value to three significant digits.
0.24936 → 0.249
0.65383 → 0.654
0.346517 → 0.347

When calculating with approximate numbers (either with or without a calculator), the precision required is determined by the context or data presented. The general rule is that the final result cannot be more precise than the least precise decimal number in the data or context.

Example
Here is a set of measurements from a scientific instrument.
3.41, 4.63, 5.789, 6.813
Find the mean, or arithmetic average.
\[
\frac{3.41 + 4.63 + 5.789 + 6.813}{4} = 5.1605
\]

Within the context of the problem, the mean result of the instrument readings should be expressed with only three significant digits: 5.16

Two instructional aids used in the Basic- and Developing-Level decimal sections are also applicable in this section. One is the 10 x 10 grid, which can be used to demonstrate the values of decimal numbers, including repeating decimal numbers.
10 x 10 Grid Strategy

Show the value 0.3 on a 10 x 10 grid.
A way to express this decimal number is \( \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots \)

Show the value 0.86 on a 10 x 10 grid.
A way to express this decimal is \( \frac{8}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \ldots \)

The number line is another good instructional aid for demonstrating values of decimal numbers. To show decimal locations more accurately, you can construct special number lines.

Number Line Strategy

Example
Locate the decimal number 0.03 on a number line.
Rather than construct a number line from 0 to 1, construct a number line from 0 to 0.1.

0 \hspace{1cm} 0.1
Benchmark 0.05 on this number line.

Now locate 0.03 to the left of the 0.05 benchmark.

**Example**
Locate the decimal number 0.008 on the number line.
Construct a number line from 0 to 0.01.

Benchmark the value 0.005 on the number line.

Now locate 0.008 to the right of the benchmark 0.005.

For advanced-level learners, most decimal number exploration should be clearly aligned with specific data or contexts. Working with complex decimal numbers without a context is not meaningful. Many science experiments require a clear understanding of precision, significant digits, and accurate rounding, so it is a natural transition to explore complex decimal numbers in a scientific context. Learners will often need to use a calculator in these situations, and as noted earlier, understanding precision, significant digits, and accurate rounding goes hand-in-hand with using a calculator.

**Percent Strategies**
At the advanced level, instruction in percents should focus on two major areas. The first covers the three elements of percents: the percentage, the rate (or percent), and the base. Learners should become confident determining all three of these.
**Examples**

**Finding the percentage**
What dollar value is 13% of $212?

\[
212 \times \frac{13}{100} = 212 \times 0.13 = $27.56
\]

Use benchmarks of 10%, 20%, 50%, and 80% to check reasonable results. Does this result seem reasonable? Yes, since the result is very close to 10% of $212.

**Finding the rate, or percent**
What is the percent discount if you received $52 off the price of $212?

\[
\frac{52}{212} = 0.245283 \text{ when using a calculator to solve this division.}
\]

However, using the procedures for rounding off to significant digits, the percent will be 25%.

**Finding the base**
You received $42 off an electric guitar purchase. You were given a discount of 24%.

What was the original price of the guitar before the discount?

\[
\frac{42}{0.24} = $175. \text{ The original price was } $175.
\]

The second area of instruction is teaching the procedure for solving all three percent situations given above. Learners should consistently use this same procedure, which involves a proportion. A proportion is a mathematical expression of two ratios that name the same number. A proportion can be as simple as \(\frac{2}{3} = \frac{4}{6}\). Both of the ratios, or fraction numbers, express the same value. When working with percents, however, the proportions will involve percentage, rate, and base. Here is a proportion procedure for solving all percent situations:

\[
\frac{\text{percentage}}{\text{base}} = \frac{\text{rate (percent)}}{100}
\]

When asking learners to solve percent problems, provide two of the three parts (percentage, rate, base). Learners will use a combination of multiplication and division to find the answer. This procedure is a consistent and efficient method for solving all percent situations.

**Using the proportion method to solve percent situations**

**Finding the percentage**
What dollar value is 13% of $212?

\[
\frac{\text{percentage}}{212} = \frac{13}{100}
\]

\[
\text{percentage} = \frac{(13 \times 212)}{100} = 13 \times 212
\]

\[
\text{percentage} = \frac{(13 \times 212)}{100} = $27.56
\]
**Finding the rate, or percent**

What is the percent discount if you received $52 off the price of $212?

\[
\frac{52}{212} = \frac{\text{rate}}{100}
\]

\[
52 \times 100 = 212 \times \text{rate}
\]

\[
\frac{(52 \times 100)}{212} = \text{rate}
\]

25% = rate

**Finding the base**

You received $42 off an electric guitar purchase. You were given a discount of 24%.

What was the original price of the guitar before the discount?

\[
\frac{42}{\text{base}} = \frac{24}{100}
\]

\[
42 \times 100 = \text{base} \times 24
\]

\[
\frac{(42 \times 100)}{24} = \text{base}
\]

$175 = \text{base}

This proportion procedure lends itself easily to the use of a calculator.

The instructional strategy of first using the hands-on mode followed by the diagram mode and then the abstract or symbolic mode still applies for teaching percents at the advanced level. It's hard to find appropriate hands-on materials for percents at this level, though, so the pictorial mode is usually the starting point. The 10 x 10 square grid used for decimal number strategies can be applied directly to percent situations. Any percent can be shown on the 10 x 10 square grid. The same instructional strategies used for the decimal numbers can be used for percents.

**Example**

Show 33 \(\frac{1}{3}\)% on a 10 x 10 grid.
Show $66\frac{2}{3}\%$ on a 10 x 10 grid.

The 10 x 10 square grid keeps the focus on the meaning of percent—parts per hundred.

Many learners at the advanced level will use a calculator to find results for percent situations. Be sure that they can set up and calculate results using the proportion method, however, so that they fully understand the different situations involving percents.

**Informal Diagnostic Strategies**

It might be assumed that all misunderstandings and errors regarding fraction numbers, decimal numbers, and percents have been eliminated by the time learners reach the advanced level. Anecdotal evidence from classroom teachers does not support this assumption, however. Teachers would be wise to use the informal paradigm detect, diagnose, prescribe, assess, and evaluate even at the advanced level to diagnose and correct misunderstandings.

The paradigm is a cycle, which starts when you detect an error pattern in the learner’s thinking. The next step, diagnose (discovering what specific conceptual or procedural error is occurring), is probably the most important. In the section below, you’ll find a discussion of common learner
errors to help you make diagnoses. After determining the error, you can prescribe specific intervention activities. Finally, be sure to assess the results and evaluate whether the misunderstanding has been corrected. If the misunderstanding still exists, start the cycle again at the diagnostic step; perhaps the learner is having difficulty with a different aspect of the procedure or concept, or if not, perhaps the learner needs more practice working with these fraction, decimal, and percent situations.

**Common Errors**

Here are some common errors you may diagnose among your learners as they work with fraction and decimal numbers at this level. Often confusion about the roles of the numerator and denominator leads to misunderstandings about adding or subtracting fraction numbers. Learners may incorrectly apply the procedure for rounding off to significant digits, or misunderstand the concept of significant digits. Another common error occurs when learners do not accurately express the proportion for solving percent situations.

**Example**

Adding two fraction numbers with like denominators

\[
\frac{3}{7} + \frac{2}{7} = ?
\]

Incorrect response: \(\frac{5}{14}\)

Incorrect reasoning: Add the numerators and add the denominators.

Correct response: \(\frac{5}{7}\)

A quick drawing of a number line with the two fractions marked should convince the learner that \(\frac{5}{14}\) is not a reasonable answer, since \(\frac{3}{7}\) and \(\frac{2}{7}\) together are more than \(\frac{1}{2}\). Review the concept of denominator as the total number of parts in the whole, and remind learners that you cannot add or subtract parts unless they come from wholes with the same number of parts.

**Example**

Subtracting two fraction numbers with unlike denominators

\[
\frac{4}{5} - \frac{3}{25} = ?
\]

Incorrect response: \(\frac{1}{20}\)

Incorrect reasoning: Subtract the numerators and subtract the denominators.

A quick drawing of a number line with the two fractions on the number line should convince the learner that \(\frac{1}{20}\) is not a reasonable answer since the difference of these fraction numbers is greater than \(\frac{1}{2}\). Review the concept of denominator as the total number of parts in the whole, and remind learners that you cannot add or subtract parts unless they come from wholes with the same number of parts.
Example
Given this instrument reading, round off to three significant digits: 0.351487

Incorrect response: 0.352
Incorrect reasoning: Use all the digits to the right of 1 to find the response. The 7 digit influences the 8 digit, making it 9, and the 9 digit influences the 4 digit, which becomes 5. The 5 will change the 1 to 2.

Correct reasoning: The 4 digit is the only digit that has an influence. Therefore, the correct response is 0.351.

Example
Jan claims she is right 100% of the time. Is Jan bragging, or what?

Incorrect response: Jan is not bragging; she is right 100% of the time.
Incorrect reasoning: misunderstanding 100% as less than the whole. (No one can be right all the time! Maybe Jan thinks she is right 100% of the time, but that's impossible.)

Example
At Haywood school, 80 athletes tried out for the track team. Only 20 athletes were chosen. What percent of the athletes trying out did not make the team?

Incorrect response:
\[
\frac{20}{80} = \frac{rate}{100} \\
\frac{(20 \times 100)}{80} = rate \\
25\% = rate
\]
Incorrect reasoning: The learner uses the number of athletes chosen for the team (20), not those who weren't chosen (60).

Correct reasoning:
\[
\frac{60}{80} = \frac{rate}{100} \\
75\% = rate
\]

Example
If 30% of the 70 items on sale were sold today, how many items were sold?

Incorrect response:
\[
\frac{30}{70} = \frac{N}{100} \\
\frac{(30 \times 100)}{70} = N
\]
Using significant digits and rounding off N = 43
Incorrect reasoning: The learner sets up the proportion incorrectly, solving for the percent rather than the percentage. Also, 43 is not a reasonable result since it is over 50% of 70 items, and we are looking for only 30%.

The correct proportion should be
\[
\frac{\text{percentage}}{70} = \frac{30}{100} \\
\text{percentage} = \frac{(70 \times 30)}{100} = 21 \text{ items}
\]
Advanced-Level Fraction Numbers

These errors are some of the most common you will see among your learners, but you'll encounter many others as well. That is why we recommend applying the informal diagnostic cycle to enhance learners’ success as they work with fraction numbers, decimal numbers, and percents.

Tips for Success

These instructional tips will help your learners grasp complex fraction numbers, decimal numbers, and percents. Use them as starter ideas and expand upon them as appropriate for your learning environment.

- Before introducing addition and subtraction with fraction numbers, make sure learners are very confident in their fraction number sense.
- Use geometric models to reinforce understanding of complex fraction numbers, decimal numbers, and percents. Two common geometric models are the number line and the 10 x 10 grid.
- Build to success in adding or subtracting fraction numbers: introduce the operations using simple fraction numbers with like denominators first.
- Consistently remind learners to use benchmarks to estimate and determine reasonable answers for fraction, decimal, and percent situations.
- Create activities involving decimal numbers and percents based on learners’ experiences.
- Give learners plenty of practice identifying significant digits for decimal numbers and percents based on the context of the situation.
- Give learners plenty of practice rounding off decimal numbers and percents as appropriate for the situation.
- Check for understanding of the different elements of working with percents (rate, percentage, and base) before introducing more complex percent situations.
- Periodically review the proportion method for solving percent situations.
- Use the instructional mode sequence: concrete to pictorial to abstract, if appropriate. At least use the sequence from pictorial to abstract at this level for most situations.
Protocols

To successfully teach fraction numbers, decimal numbers, and percents, teachers must monitor learners' progress during practice and intervene between the practices as needed. The following protocols can be used as intervention activities. Each protocol identifies a specific topic and presents pre-assessment information, the appropriate development level, a sequence of actions, and an evaluation strategy. You will also find suggestions for adapting the protocol for special needs, gifted, and English language learners.

The protocols are designed to take 20 to 30 minutes. The Launch section, which describes any preparation needed and instructions for teachers, may take 5 minutes; the Explore section, which describes what the students will do, 15 minutes; the Closure section 2 minutes and the Assessment and Evaluation sections together another 5 minutes. Of course, the protocols are quite flexible; you may modify them as needed for the most effective use in your particular learning environment. The time frame of each protocol can easily be adapted to any learning environment as well.

The protocols in this Advanced-Level section focus on the addition and subtraction of fraction numbers and on decimal numbers and percents. Some are appropriate for small groups of two to four learners, and some involve the entire class. Many can be completed by an individual learner. In the case of the small groups, depending on the protocol, you may be able to present the protocol as either a learning activity or as a competitive game. Sometimes learners are stimulated by the gaming aspect of learning; other times, the competitive nature of winning or losing can detract from the learning environment. All the protocols require minimal materials. You may need to create some of the materials but they are simple to construct; reproducible forms of specific instructional aids are provided in the appendix.
To find interventions suitable for your learners, refer to the matrix below.

### Advanced-Level Protocols

<table>
<thead>
<tr>
<th>Title</th>
<th>Topic</th>
<th>Instructional Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction Number Comparison</td>
<td>comparing addition of fraction numbers to benchmarks</td>
<td>individual, small group</td>
</tr>
<tr>
<td>Pick Your Fraction</td>
<td>adding and subtracting fraction numbers with like denominators</td>
<td>whole group, individual</td>
</tr>
<tr>
<td>Numerator and Denominator Challenge</td>
<td>properties related to the addition of fraction numbers</td>
<td>small group</td>
</tr>
<tr>
<td>Stack Them Up</td>
<td>converting fraction numbers to decimal numbers</td>
<td>small group</td>
</tr>
<tr>
<td>Build a Biggie</td>
<td>constructing decimal values</td>
<td>whole group, small group</td>
</tr>
<tr>
<td>Decimal Potpourri</td>
<td>estimating decimal values</td>
<td>whole group, individual</td>
</tr>
<tr>
<td>Heads &amp; Tails</td>
<td>expressing data with percents</td>
<td>small group</td>
</tr>
<tr>
<td>Draw a Percent</td>
<td>integration of art and mathematics with percents</td>
<td>whole group, individual</td>
</tr>
</tbody>
</table>
Fraction Number Comparison

**Topic:** Comparing addition of fraction numbers to benchmarks  
**Level:** Advanced  
**Instructional Mode:** Individual, small group (two learners)  
**Time:** 20–30 minutes

### Pre-assessment
- Check for learners’ understanding of using benchmarks to estimate.

### Materials
- Fraction Number Cards  
  *(Reproducible cards are provided in the appendix.)*

### Launch
Explain the activity by demonstrating the procedures.
- Draw two Fraction Number Cards from the deck, for example, $\frac{1}{3}$ and $\frac{1}{4}$.
- Consider the result of adding the two fractions.
  - First, will the sum of the two fractions be greater than 1, equal to 1, or less than 1? (less than 1)
  - Second, will the sum of the two fractions be greater than $\frac{1}{2}$, equal to $\frac{1}{2}$, or less than $\frac{1}{2}$? (greater than $\frac{1}{2}$)
- Discuss strategies for benchmarking when adding two fractions.

### Explore
- Place all the Fraction Number Cards face down in a grid pattern.
- The learner picks up two of the Fraction Number Cards.
- This learner estimates the sum of the two fractions.
  - Greater than, less than, or equal to 1?
  - Greater than, less than, or equal to $\frac{1}{2}$?
- This learner declares the benchmarked comparison result.
  - This learner keeps these two Fraction Number Cards.
- If two learners are doing the activity together, the second learner follows the same procedure.
- Continue until all the Fraction Number Cards have been used.

### Closure
- Have the learner choose a pair of Fraction Number Cards from his or her pile.
- Have the learner demonstrate the process of comparison as a review.
**Assessment**

Choose any two Fraction Number Cards.

Ask the learner to estimate the sum of two fractions using both 1 and $\frac{1}{2}$ as the benchmarks.

**Evaluation**

Success will be shown when learners can readily and correctly determine an estimated answer for the sum of two proper fractions using benchmarking.

<table>
<thead>
<tr>
<th>Adaptations for Various Learners</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ELL</strong></td>
</tr>
<tr>
<td>• Explain the terms <em>less than</em>, <em>greater than</em>, and <em>equal to</em> in this context of benchmarking.</td>
</tr>
<tr>
<td><strong>Special Needs</strong></td>
</tr>
<tr>
<td>• Remove the majority of the larger denominator Fraction Number Cards.</td>
</tr>
<tr>
<td>• Learners complete the activity using only 1 as a benchmark.</td>
</tr>
<tr>
<td><strong>Gifted</strong></td>
</tr>
<tr>
<td>• Have two learners play the activity as a game.</td>
</tr>
<tr>
<td>• The first learner selects two Fraction Number Cards from the grid and estimates the sum using benchmarking.</td>
</tr>
<tr>
<td>• The second learner does the same.</td>
</tr>
<tr>
<td>• The learner with the larger estimate wins the round and receives all four Fraction Number Cards.</td>
</tr>
<tr>
<td>• Continue play until all the Fraction Number Cards are used.</td>
</tr>
<tr>
<td>• The winner is the learner with the most Fraction Number Cards.</td>
</tr>
</tbody>
</table>
Pick Your Fraction

**Topic:** Adding and subtracting fraction numbers with like denominators  
**Level:** Advanced  
**Instructional Mode:** Whole group, individual  
**Time:** 20–30 minutes

**Pre-assessment**
- Check that learners have experience with Fraction Number Cards and Fraction Circles. If not, spend time exploring each instructional aid.

**Materials**
- Fraction Number Cards and Fraction Circles  
  (*Reproducible cards and circles are provided in the appendix.*)

**Launch**
- Ask learners to help you arrange the Fraction Number Cards and the Fraction Circles by like denominators (same color stacks).
- Select two Fraction Number Cards with like denominators from the set.
- Ask learners to suggest what Fraction Circles to use to find the sum, and then find the result.

**Explore**
- Begin the activity by limiting the possible sums to less than 1 or equal to 1. Later, if desired, allow the possible sums to be greater than 1.
- Ask learners to select two Fraction Number Cards with like denominators. Consider adding these two fractions. Determine whether the result is less than 1, equal to 1, or greater than 1.
- If less than 1, record the operation by writing the fraction expressions and determining the sum. If not less than 1, pick another two Fraction Number Cards.
- Continue until sufficient pairs of fractions have been selected.
- Now repeat the activity by *subtracting* the smaller fraction from the larger fraction.
- Use the Fraction Circles for help adding or subtracting fractions with like denominators.

**Closure**
- Ask the learner to select two Fraction Number Cards with like denominators.
  - If the sum or difference is less than 1, ask the learner to state the result.
  - Discuss the strategies for adding or subtracting two fractions with like denominators.
## Assessment

Have the learner select two Fraction Number Cards with like denominators.

Have the learner estimate, without using the Fraction Circles, whether the sum (or difference) of these two fractions is less than 1.

If the sum (or difference) is less than 1, determine the result. If not less than 1, select again.

## Evaluation

Success will be shown when the learner can add or subtract with confidence two fractions with like denominators without instructional aids.

### Adaptations for Various Learners

**ELL**
- Explain the term *like denominator*.

**Special Needs**
- Remove the majority of the Fraction Number Cards to make a smaller set.
- Remove the corresponding Fraction Circles to make a smaller set.
- Be sure learners explore only fractions that have a sum of less than 1.

**Gifted**
- Allow any combination of Fraction Number Cards with like denominators to be used.
- The sum may be greater than 1 (e.g., \( \frac{4}{6} + \frac{3}{6} = \frac{9}{6} \)).
- Have learners express the result in simplified form (e.g., \( \frac{9}{6} \) as \( \frac{3}{2} \)).
- Have learners express the result in a mixed-number format (e.g., \( 1 \frac{1}{2} \)).
Numerator and Denominator Challenge

**Topic:** Properties related to the addition of fraction numbers

**Instructional Mode:** Small group (two to four learners)

**Level:** Advanced

**Time:** 20–30 minutes

**Pre-assessment**
- Check for knowledge of the developing-level protocol Numerator and Denominator.

**Materials**
- Two number cubes with these values on the faces: 1, 2, 3, 4, 6, 8
- A Rectangular Recording Sheet for each group (A reproducible copy is provided in the appendix.)
- A different color marker or crayon for each learner in the group

**Launch**
Explain the procedures for the activity or game:
- Each learner rolls the two number cubes. The numbers shown must be formed into a proper fraction with one number as the numerator and the other number the denominator. For example, if the values 3 and 4 were rolled, the proper fraction would be \( \frac{3}{4} \).
- Next the learner represents the proper fraction in another form, and then shades that on the Rectangular Recording Sheet. For example, the fraction \( \frac{3}{4} \) can be represented by \( \frac{1}{4} \) of one rectangle and \( \frac{3}{4} \) of another rectangle. For this activity, the learner cannot simply shade \( \frac{3}{4} \) of the rectangle that is divided into fourths. The only exceptions are when \( \frac{1}{6} \) and \( \frac{1}{8} \) are created by the roll of the number cubes.
- The object is to have the maximum number of parts colored on the group Rectangular Recording Sheet.

**Explore**
- Determine which learner starts by rolling one number cube; the learner who rolls the largest value begins the activity.
- The learner rolls both number cubes and decides how to color the proper fraction with the given restrictions.
- Learners take turns rolling the number cubes and recording the appropriate values.
- The activity continues until all the rectangles have been colored.
Closure

- If this is an activity, each learner can count how many fractional parts of the rectangles he or she colored.
- If this is a game, use the following scoring scheme:
  - Five points for each rectangle completely colored by the learner.
  - Two points for a rectangle that is more than half colored by the learner.
  - Zero points for all rectangles that are less than half colored by the learner.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask learners to roll both number cubes.</td>
<td>Success is shown as learners do this activity with more confidence and accuracy, including being more creative using the fractional parts to color the rectangles.</td>
</tr>
<tr>
<td>Have learners form the proper fraction.</td>
<td></td>
</tr>
<tr>
<td>Ask learners to name a different form of the fraction number and to describe one way they could color the rectangles to represent it.</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

**ELL**
- Explain the meaning of proper fraction, fractional parts, rectangle, and rectangular shape.

**Special Needs**
- Only color one form or expression shown on the number cubes on one rectangle.
- Do not use the scoring scheme; instead, have learners count how many fractional parts they colored.

**Gifted**
- Encourage more creative ways to color the rectangles.
- Allow more than two forms of a fraction to be colored (e.g., for the fraction $\frac{5}{6}$, learners could shade or cover $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{1}{3}$).
Stack Them Up

**Topic:** Converting fractions to decimal numbers  
**Level:** Advanced  
**Instructional Mode:** Small group (two learners)  
**Time:** 20–30 minutes

**Pre-assessment**
- Check for understanding of how to express a fraction number as a decimal number.

**Materials**
- calculator
- strips of blank paper (approximately 1 inch wide and 3 inches long)

**Launch**
Demonstrate the basic activity:
- Choose any two different numbers from 1 through 9, for example 1 and 4.
- Make a proper fraction with these selected numbers: $\frac{1}{4}$
- Convert this fraction into the corresponding decimal number using a calculator: 0.25
- Record the information on a blank strip of paper. (Take the decimal number to the hundredths place.)

\[
\frac{1}{4} = 0.25
\]

Explain the stacking part of the activity:
- After the first student has created fraction-decimal equivalents, the next student does the same.
  Pick two different numbers, 1 and 3, to form $\frac{1}{3}$. Convert to the corresponding decimal number: 0.33

\[
\frac{1}{3} = 0.33
\]

- Then learners compare the decimal numbers. If the new one is larger than the first, make a stack with the paper strips:

\[
\begin{align*}
\frac{1}{3} & = 0.33 \\
\frac{1}{4} & = 0.25
\end{align*}
\]
• If the second decimal number is smaller than the first, the stack stops and the learners start a new stack.

**Explore**
• Learners take turns picking two numbers to form a fraction, converting it to the corresponding decimal number, and adding to the stack or starting a new one.
• The aim is to create a tall stack of strips. See how high the stack can go!

**Closure**
• Discuss with learners the strategy of picking numbers to make a tall stack.
• Remind learners to use their knowledge of the magnitude of fractions in picking their two numbers.

<table>
<thead>
<tr>
<th><strong>Assessment</strong></th>
<th><strong>Evaluation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Write one fraction each on three different strips, without the corresponding decimal number.</td>
<td>Success will be shown when the learners can confidently convert the fraction to the corresponding decimal number and correctly order a set of fraction/decimal number strips from smallest to largest.</td>
</tr>
<tr>
<td>Ask learners to find and record the corresponding decimal number and then order the strips from smallest to largest in a stack.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Explain the words *convert* and *stack*.
• Show how to form a stack.

**Special Needs**
• Review how to use a calculator to find the corresponding decimal number from a given fraction.
• Premake a set of strips with the fraction number and the corresponding decimal number. The learner can find the correct strip after selecting the two numbers.
• Encourage making the stacks no more than two or three levels high.

**Gifted**
• Have learners select numbers that are not in simplified form (e.g., $\frac{6}{9}$).
• On the strip of paper have the learner express the original fraction, the simplified form of the fraction, and the corresponding decimal number.
• Order the stacks from largest to smallest as well as from smallest to largest.
Build a Biggie

**Topic:** Constructing decimal values

**Instructional Mode:** Whole group, small group (two to four learners)

**Level:** Advanced

**Time:** 20–30 minutes

### Pre-assessment
- Check for understanding of tenths, hundredths, and thousandths place values for decimal numbers.

### Materials
- a set of 10 index cards, each with a single digit from 0 to 9 written on it

### Launch
- Remove the zero digit card from the set to begin the activity. Shuffle and place cards in a deck face down.
- Explain the procedures for completing the activity:
  - Begin the activity by building a three-digit decimal number (tenths and hundredths).
    - 0. ______ ______
  - The goal is to build the largest decimal value possible with the drawn cards.
  - Draw a digit card from the set of cards (e.g., 6).
  - Decide where to record the 6 digit within the two-digit decimal. It could be recorded either as 0.6 or 0.06. To demonstrate, write 6 in the tenths slot.
  - Without replacing the 6 digit card, draw another card (e.g., 8).
  - Now the 8 must be recorded in the open slot, which is the hundredths place. The result is 0.68.
  - Clearly, the digits must be recorded immediately after drawing the digit card to introduce elements of chance and strategy to this activity.
  - However, if the 6 digit had been recorded in the hundredths place as 0.06 and the 8 digit had been recorded in the tenths place, the result, 0.86, would have been larger than 0.68.
  - Show another example to build learners’ confidence in completing the activity.

### Explore
- This activity can be done as a whole group with all learners deciding where to place the drawn digits. A learner or teacher could draw each digit card; the other learners construct the decimals.
- Begin the activity with the 1 through 9 digit cards and limit the decimals to tenths and hundredths:
  - 0. ______ ______
• Do this activity for at least five rounds.
• Later add the zero digit card to the set and increase the decimal places to the thousandths:
  0. ______ ______ ______
• Continue this activity for at least five more rounds.

**Closure**
• Discuss the strategies used to construct the largest possible decimal number.
• Remind the learners of the importance of place value and the impact of the digit magnitude on their location choice when constructing a decimal number.
• Explore the strategies for placing the 5 digit in a decimal value.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw a digit card.</td>
<td>Success will be shown when learners can quickly and confidently make a decision where to place a digit in a decimal in order to potentially construct a large value decimal.</td>
</tr>
<tr>
<td>Have the learner place the digit in a decimal 0. _____ _____</td>
<td></td>
</tr>
<tr>
<td>Draw another digit card.</td>
<td></td>
</tr>
<tr>
<td>Have the learner place this digit in the decimal value.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Review the meaning of the terms *tenths, hundredths, thousandths*, and *larger value decimal number*.

**Special Needs**
• Begin the activity with only four cards, 1, 3, 7, and 9, to build confidence in finding a large value decimal number.
• Later add the remaining cards, except zero, to complete the activity.
• Have learners use the cards to construct the decimal number rather than writing the decimal values in blank slots.

**Gifted**
• After playing Build a Biggie, play Keep It Small, where the aim is to create the smallest decimal value.
• Play Build a Biggie or Keep It Small using whole numbers along with decimal numbers:
Decimal Potpourri

**Topic:** Estimating decimal number values  
**Level:** Advanced  
**Instructional Mode:** Whole group, individual  
**Time:** 20–30 minutes

### Pre-assessment
- Check for understanding of tenths, hundredths, and thousandths place values in decimal numbers.

### Materials
- a calculator for checking estimations

### Launch
- The Decimal Potpourri is a collection of activities that involve estimating decimal numbers using benchmarks 0, 0.5, and 1.0, estimating decimal numbers between two given decimal numbers, and estimating possible decimal values.

### Explore

#### Estimating with benchmarks 0, 0.5, and 1.0
- Which decimal numbers are closest to 0, 0.5, and 1.0?
- Begin with tenths: 0.4, 0.8, 0.1, 0.6
- Continue with hundredths: 0.45, 0.89, 0.15, 0.69
- Continue with thousandths: 0.459, 0.891, 0.156, 0.693
- Construct other decimal value sets and benchmark closest to 0, 0.5, and 1.0.

#### Estimating a decimal value between two given decimal values
- Have learners name a possible decimal value, in tenths, between 0.3 and 0.8. (possible result: 0.6)
- Ask learners to name a possible decimal value, in hundredths, between 0.13 and 0.45. (possible result: 0.27)
- Ask learners to name a possible decimal value, in thousandths, between 0.236 and 0.275. (possible result: 0.249)
- Construct other decimal value pairs and find a possible decimal between these pairs of decimals.

#### Estimating possible decimal values
- What is missing? 5 x ____ = 96?
  - Guess a decimal value in tenths (do not divide) and check your guess with a calculator.  
    5 x guess = ? (close to 96?) Possible guesses might include 18.6, then 19.5.
  - Continue guessing until 96 is obtained. (correct answer: 19.2)
• What is missing? 6 x _____ = 107
  − Guess a decimal value in hundredths (do not divide) and check your guess with a calculator.  
    6 x guess = ? (close to 107?)  
  − Continue guessing until 107 is almost obtained.
• Construct other “What is missing?” expressions, guess a decimal value, and check the guess with a calculator.

**Closure**
• Review the potpourri of activities for estimating with decimal numbers.
• Discuss some of the strategies used to make an appropriate guess.

<table>
<thead>
<tr>
<th><strong>Assessment</strong></th>
<th><strong>Evaluation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Have learners construct a decimal value with a thousandths place (e.g., 0.149).</td>
<td>Success will be shown when learners can accurately construct the decimal number, determine the closest benchmark, and name a decimal number between this decimal and 0.500.</td>
</tr>
<tr>
<td>Have learners determine the closest benchmark to this decimal value.</td>
<td></td>
</tr>
<tr>
<td>Have learners find a decimal value between this decimal number and 0.500.</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**
• Review the terms benchmark, closest to, between two decimals, and missing decimal.

**Special Needs**
• Limit the activity to decimal values involving tenths.
• Later you might expand to decimal values involving hundredths.
• Partner the learner with another learner to check the guesses with a calculator.

**Gifted**
• Introduce more benchmarks: 0.10, 0.25, 0.50, 0.75, and 0.90
• Using the expanded list, have learners complete the benchmark activity.
• Have learners find more than one decimal value between two given decimal numbers.
• Add complexity to the missing decimal situations (e.g., 18 x _____ = 126.5)
Heads & Tails

Topic: Expressing data with percents  
Level: Advanced
Instructional Mode: Small group (two to four learners)  
Time: 20–30 minutes

Pre-assessment
• Check knowledge of expressing a penny as a percent of a dollar (1% of a dollar).

Materials
For each small group
• a bag of 100 pennies
• a felt or cloth square, 12” x 12”
• Heads & Tails Recording Sheet  
(A reproducible copy is provided in the appendix.)

Launch
• Discuss the two ways a penny can land: heads or tails.
• Demonstrate how to use the felt square to contain the pennies for accurate counting of the number of heads and tails:
  − Carefully empty the bag of pennies onto the felt or cloth square.
  − Place all the heads on one half of the felt square.
  − Place all the tails on the other half of the felt square.
  − Arrange the pennies in rows and columns to aid in counting them.

Explore
• Give each small group a bag of 100 pennies and a felt square.
• Construct a simple recording sheet (or use the reproducible form provided in the appendix.)
• The bag of 100 pennies is carefully distributed on the felt square.
• Count the number of heads.
• Count the number of tails.
• Record these values on the recording sheet.
• Determine the percent of heads and the percent of tails for each trial.
• Complete at least six trials counting and recording the distribution of heads and tails pennies.
**Closure**

- Do you always need to count the number of tails if you have already counted the number of heads for a trial? Why or why not?
- Ask learners to show their reasoning by demonstrating with the bag of pennies.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ask learners to distribute the 100 pennies on the felt square.</td>
<td>Ask: If there are 100 students and 43 of them are girls, what percent of the students are boys? Success will be shown when the learners are confident that they can express the percent when the base is 100.</td>
</tr>
<tr>
<td>Have them count the number of tails. What is the percent of tails?</td>
<td></td>
</tr>
<tr>
<td>Have them determine the number of heads and the percent of heads.</td>
<td></td>
</tr>
<tr>
<td>What is the total of the percents for both heads and tails for this trial?</td>
<td></td>
</tr>
</tbody>
</table>

**Adaptations for Various Learners**

**ELL**

- Review the value of a penny in the money system.
- Review that 100 pennies is equivalent to one dollar, thus one penny is $0.01 or 1% of a dollar.

**Special Needs**

- Review the value of a penny in the money system.
- Review that 100 pennies is equivalent to one dollar, thus one penny is $0.01 or 1% of a dollar.
- Make the felt square larger (e.g., 24” x 24”) for ease in counting heads and tails.
- Partner the learner with another who has accurately completed the activity.

**Gifted**

- Have learners plot some of the class data using these methods:
  - Box plot of heads vs. tails per trial
  - Bar graph of heads vs. tails per trial
  - Stem and leaf plot of heads vs. tails per trial
- Have learners use computer software to collect the data and display the results.
Draw a Percent

**Topic:** Integration of art and mathematics with percents  
**Level:** Advanced  
**Instructional Mode:** Whole group, individual  
**Time:** 20–30 minutes

**Pre-assessment**
- Check understanding that the 10 x 10 square grid, or 100 squares, is a representation of percent (per hundred).

**Materials**
- a variety of 10 x 10 square grids of different size squares  
  *(A reproducible form is provided in the appendix.*)  
- colored markers or pencils

**Launch**
- Demonstrate with a 10 x 10 grid how to make an art image using a percent value.  
  - For example, show this drawing of 26%:

![10 x 10 grid with 26% shaded](image)

**Explore**
- Have the learners determine the percent they want to display on a 10 x 10 grid as an art image.  
- Have them color or mark the 10 x 10 grid with their percent and title the art image as the percent value.  
- Allow the learners to make many different art images using different percent values.  
- Display the art images in the classroom for other learners to view.
Closure
• Have other learners check that the correct percent value is shown on the 10 x 10 art image.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Select one of the art images.</td>
<td>Success will be shown when the learner can quickly and confidently shade any given percent and also identify what percent is not shaded.</td>
</tr>
<tr>
<td>Determine if the correct percent value is displayed on the 10 x 10 grid.</td>
<td></td>
</tr>
<tr>
<td>Ask: If X% (the selected art image) is shaded, what percent of the art image is not shaded?</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations for Various Learners

ELL
• Explain what is meant by an art image.

Special Needs
• At the beginning of the activity, use only percents less than 20%.
• Later, increase the magnitude of the percents to at least 50%.
• Use only the 10 x 10 grids with larger squares for ease in coloring or marking.

Gifted
• Require learners to display an art image containing no whole squares shaded for the selected percent.
• Allow percents that are not whole numbers (e.g., 33 \(\frac{1}{3}\)% or 66 \(\frac{2}{3}\)%).
Appendix

144 Reproducible Forms

144 Fraction Circles
154 Fraction Number Cards
158 Interlocking Circles
160 Close Is Just Fine Recording Sheet
162 10 x 10 Grid Sheet
163 10 x 10 Grid Sheet
164 Match Them All Score Sheet
165 Match Them All Fraction/Percent Reference Chart
166 Marker Cover Up Game Board
167 Rectangular Recording Sheet
168 Decimal Number Cover Game Board
169 Decimal Number Cards
170 Label Cards
171 Decimal Name Cards
172 Grid Percent Matrix Sheet
173 Heads & Tails Recording Sheet
174 10 x 10 Grid Sheet for Draw a Percent

175 References
Fraction Circles

To create your own set of Fraction Circles, photocopy each of the following pages on a different color card stock so that each fraction in the set is a different color. Then cut out each fraction piece.
Fraction Number Cards

To create your own set of Fraction Number Cards, photocopy the following pages on card stock. Use the same color paper for the entire set. Then cut out each Fraction Number Card.
<table>
<thead>
<tr>
<th>6/1</th>
<th>4/3</th>
<th>2/1</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/2</td>
<td>5/1</td>
<td>3/1</td>
</tr>
<tr>
<td>6/3</td>
<td>5/2</td>
<td>3/2</td>
</tr>
<tr>
<td>6/4</td>
<td>5/3</td>
<td>4/1</td>
</tr>
<tr>
<td>6/5</td>
<td>5/4</td>
<td>4/2</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>
Interlocking Circles

To create your own set of Interlocking Circles, make two copies of the next page, each on a different color card stock (for example, red and blue). Then cut out each Interlocking Circle and its radius.
## Close Is Just Fine Recording Sheet

<table>
<thead>
<tr>
<th>Drawing of the Object</th>
<th>Close Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Drawing of the Object</td>
<td>Close Measure</td>
</tr>
<tr>
<td>-----------------------</td>
<td>--------------</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10 x 10 Grid Sheet
10 x 10 Grid Sheet
# Match Them All Score Sheet

<table>
<thead>
<tr>
<th>Name of Learner</th>
<th># of Two-Value Matches (3 points each)</th>
<th># of Three-Value Matches (6 points each)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Match Them All Fraction/Percent Reference Chart

<table>
<thead>
<tr>
<th>Fraction/Percent</th>
<th>0.25</th>
<th>0.33</th>
<th>0.50</th>
<th>0.67</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td></td>
<td>25%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>33%</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50%</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>67%</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>75%</td>
</tr>
</tbody>
</table>
## Marker Cover Up Game Board

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
<td>(\frac{5}{6})</td>
<td>(\frac{3}{6})</td>
<td>(\frac{2}{5})</td>
<td>(\frac{2}{4})</td>
</tr>
<tr>
<td>(\frac{4}{5})</td>
<td>(\frac{4}{8})</td>
<td>(\frac{2}{2})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>(\frac{6}{8})</td>
<td>(\frac{3}{5})</td>
<td>(\frac{2}{6})</td>
<td>(\frac{4}{4})</td>
<td>(\frac{5}{8})</td>
</tr>
<tr>
<td>(\frac{4}{6})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{7}{8})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{5})</td>
</tr>
<tr>
<td>(\frac{2}{3})</td>
<td>(\frac{3}{8})</td>
<td>(\frac{3}{4})</td>
<td>(\frac{6}{6})</td>
<td>(\frac{2}{8})</td>
</tr>
</tbody>
</table>
## Decimal Number Cover Game Board

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.83</td>
<td>0.50</td>
<td>0.40</td>
<td>0.50</td>
</tr>
<tr>
<td>0.80</td>
<td>0.50</td>
<td>1.00</td>
<td>0.13</td>
<td>0.33</td>
</tr>
<tr>
<td>0.75</td>
<td>0.60</td>
<td>0.50</td>
<td>1.00</td>
<td>0.63</td>
</tr>
<tr>
<td>0.67</td>
<td>0.25</td>
<td>0.88</td>
<td>0.17</td>
<td>0.20</td>
</tr>
<tr>
<td>0.38</td>
<td>0.67</td>
<td>0.75</td>
<td>1.00</td>
<td>0.25</td>
</tr>
</tbody>
</table>
**Decimal Number Cards**

To create your own set of Decimal Number Cards, photocopy this page onto card stock. Then cut out each card.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.60</td>
<td>0.25</td>
</tr>
<tr>
<td>0.30</td>
<td>0.33</td>
<td>0.50</td>
</tr>
<tr>
<td>0.70</td>
<td>0.67</td>
<td>0.75</td>
</tr>
<tr>
<td>1.00</td>
<td>0.80</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>0.40</td>
</tr>
</tbody>
</table>
Label Cards

< 0.50

= 0.50

> 0.50
### Decimal Name Cards

To create your own set of Decimal Name Cards, photocopy this page onto card stock. Then cut out each card.

<table>
<thead>
<tr>
<th>Hundredths</th>
<th>Hundredths</th>
<th>Hundredths</th>
</tr>
</thead>
<tbody>
<tr>
<td>ten hundredths</td>
<td>sixty hundredths</td>
<td>twenty-five hundredths</td>
</tr>
<tr>
<td>thirty hundredths</td>
<td>thirty-three hundredths</td>
<td>fifty hundredths</td>
</tr>
<tr>
<td>seventy hundredths</td>
<td>sixty-seven hundredths</td>
<td>seventy-five hundredths</td>
</tr>
<tr>
<td>one hundredths</td>
<td>eighty hundredths</td>
<td>twenty hundredths</td>
</tr>
<tr>
<td></td>
<td>ninety hundredths</td>
<td>forty hundredths</td>
</tr>
</tbody>
</table>
### Grid Percent Matrix Sheet

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
<th>Picture</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.46</td>
<td></td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>(\frac{22}{100})</td>
<td></td>
<td>63%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td></td>
<td>0.86</td>
<td></td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
<tr>
<td>(\frac{57}{100})</td>
<td></td>
<td>13%</td>
<td>Use 10 x 10 Grid Sheet</td>
</tr>
</tbody>
</table>
# Heads & Tails Recording Sheet

<table>
<thead>
<tr>
<th>Trial</th>
<th>Number of Heads</th>
<th>Percent of Heads</th>
<th>Number of Tails</th>
<th>Percent of Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10 x 10 Grid Sheet for Draw a Percent

[Diagram of 10 x 10 grid sheets]
References


*Journal for Research in Mathematics Education*, 41(1), 20–51.


*Journal of Educational Psychology*, 93(2), 346–362.


*Learning and Instruction*, 14(1), 453–467.
About Renaissance

Renaissance® is the leader in K-12 learning analytics—enabling teachers, curriculum creators, and educators to drive phenomenal student growth. Renaissance’s solutions help educators analyze, customize, and plan personalized learning paths for students, allowing time for what matters—creating energizing learning experiences in the classroom. Founded by parents, upheld by educators, and enriched by data scientists, Renaissance knows learning is a continual journey—from year to year and for a lifetime. Our data-driven, personalized solutions are currently used in over one-third of U.S. schools and more than 60 countries around the world. For more information, visit www.renaissance.com.